University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam Solutions
January 29, 2021

Name: ____

Exam Rules:

- This exam is being administered remotely using Zoom. During the exam, you must be logged in to the assigned Zoom meeting with your your camera on, and you must be visible in the camera. Your microphone can be muted, but please leave Zoom audio ON so that the proctor can speak to you if needed.
- If you have any questions during the exam, please contact the exam proctor using the zoom chat, or call the proctor at the number given below.

  Committee Member Contact Information:

Name	Phone	email	Proctoring times
Julien Langou	xxx-xxx-xxxx	julien.langou@ucdenver.edu	10:00am-12:00pm
Yaning Liu	xxx-xxx-xxxx	yaning.liu@ucdenver.edu	12:00pm-2:00pm
Jan Mandel		jan.mandel@ucdenver.edu	

- If you need to be out of view of the camera for any reason (bathroom breaks, etc.), please let the proctor know by posting a message on the zoom chat.
- If you would like to work with a paper copy of the exam, please print it as soon as you receive it and inform the proctor that you are doing so before the exam begins. If the printer is in another room, let the proctor know.
- You may read the exam as soon as you receive it, but you may not start writing (even your name) until authorized to start writing.
- This is a closed book exam. You may not use any external aids during the exam, such as:
  - communicating with anyone other than the exam proctor (through text messages or emails, for example);
  - consulting the internet, textbooks, notes, solutions of previous exams, etc;
  - using calculators or mathematical software.
- You may use a tablet PC (such as an iPad or Microsoft Surface) to write your solutions. Alternatively, you can write your solutions on paper.
- Please begin each problem on a new page, and write the problem number and page number at the top of each page. (For example, 6-1, 6-2, 6-3 for pages 1, 2 and 3 of problem 6). If you are writing on paper, write only on one side of the paper.
- The exam will end 4 hours after it begins. At the conclusion of the exam, please email a copy of your solutions to the exam proctor. Do not leave until the proctor acknowledges that your solutions have been successfully received.

  - Your solutions need to be in a single .pdf file with the pages in the correct order. The .pdf file needs to be of good enough quality for easy grading.
  - If you cannot create a good quality .pdf file quickly, you may instead submit an imperfect scan, or even pictures of your exam, and then take more time to prepare and submit a good quality .pdf version. We will grade the better version but use the first submission to check that nothing was added or changed between versions.
  - Do not submit your scratch work.

- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded. Each problem is worth 20 points.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- <u>Notation</u>: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{n,n}$ are the vector spaces of n-tuples and $n \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- If you are confused or stuck on a problem, either ask a question or move on to another problem.

Problem	Points	Score		Problem	Points	Score
1.	20			5.	20	
2.	20			6.	20	
3.	20			7.	20	
4.	20			8.	20	
				Total	120	

Applied Linear Algebra Preliminary Exam Committee:
Julien Langou, Yaning Liu (Chair), Jan Mandel

Before starting the exam, please sign and date the following honor statement.

Honor statement: I attest that I will not cheat and will not attempt to cheat. I attest that I will not communicate with anyone while taking the exam, I will not look at notes, textbooks, previous solutions of exams, cheat sheets, etc. I attest that I will not go on the web to find solutions. If I in some way receive information during the exam that might help with the exam, I will let the proctor know.

Part I. Work **all** of problems 1 through 4.

Problem 1. Suppose $v_1, \ldots, v_m$ is linearly independent in V, $w \in V$, and $v_1 + w, \ldots, v_m + w$ is linearly dependent. Prove that $w \in \operatorname{span}\{v_1, \ldots, v_m\}$.

Solution: Since $v_1 + w, \ldots, v_m + w$ is linearly dependent, there exist scalars $a_1, \ldots, a_m$ not all zero such that

$$a_1 (v_1 + w) + \cdots + a_m (v_m + w) = 0.$$

It follows that

$$a_1 v_1 + \cdots + a_m v_m = -(a_1 + \cdots + a_m) w. \tag{1}$$

If $a_1 + \cdots + a_m = 0$, then

$$a_1 v_1 + \cdots + a_m v_m = 0w = 0$$

which contradicts the independence of $v_1, \ldots, v_m$. Thus $a_1 + \cdots + a_m \neq 0$ and we can divide (1) by it, giving

$$\begin{aligned} w &= \frac{a_1 v_1 + \cdots + a_m v_m}{-(a_1 + \cdots + a_m)} \\ &= \frac{-a_1}{a_1 + \cdots + a_m} v_1 + \cdots + \frac{-a_m}{a_1 + \cdots + a_m} v_m, \end{aligned}$$

which shows that $w \in \operatorname{span}\{v_1, \ldots, v_m\}$.

Problem 2. Suppose that V is inner product space, real or complex.

1. (10 points) Prove that if $u, v, w \in V$, then

$$\left\| w - \frac{1}{2}(u + v) \right\|^2 = \frac{\|w - u\|^2 + \|w - v\|^2}{2} - \frac{\|u - v\|^2}{4}.$$

2. (10 points) Suppose that S is a subset of V such that if $u, v \in S$ then $\frac{1}{2}(u + v) \in S$, and $w \in V$. Show that there is at most one point u in S that is closest to w, that is, such that

$$\|w - u\| \leq \|w - y\| \text{ for all } y \in S.$$

Solution:

1. Identify common terms first:

$$\begin{aligned}\left\|w-\frac{1}{2}(u+v)\right\|^2 &= \left\|\frac{1}{2}((w-u)+(w-v))\right\|^2\\ &= \frac{1}{4}\langle (w-u)+(w-v),(w-u)+(w-v)\rangle\\ &= \frac{1}{4}\left(\|w-u\|^2+\|w-v\|^2+2\operatorname{Re}\langle w-u,w-v\rangle\right)\end{aligned}$$

$$\begin{aligned}\frac{\|w-u\|^2+\|w-v\|^2}{2}-\frac{\|u-v\|^2}{4} &= \frac{1}{4}\left(\|w-u\|^2+\|w-v\|^2\right)+\\ &\quad\frac{1}{4}\left(\|w-u\|^2+\|w-v\|^2-\|u-v\|^2\right).\end{aligned}$$

It remains to show that $\|w-u\|^2+\|w-v\|^2-\|u-v\|^2 = 2\operatorname{Re}\langle w-u,w-v\rangle$. Expanding the inner products, we have

$$\begin{aligned}\|w-u\|^2+\|w-v\|^2-\|u-v\|^2 &= \left(\|w\|^2+\|u\|^2-2\operatorname{Re}\langle w,u\rangle\right)+\left(\|w\|^2+\|v\|^2-2\operatorname{Re}\langle w,v\rangle\right)\\ &\quad-\left(\|u\|^2+\|v\|^2-2\operatorname{Re}\langle u,v\rangle\right)\\ &= 2\|w\|^2+2\operatorname{Re}\left(-\langle w,u\rangle-\langle w,v\rangle+\langle u,v\rangle\right)\end{aligned}$$

$$\begin{aligned}2\operatorname{Re}\langle w-u,w-v\rangle &= 2\operatorname{Re}\left(\langle w,w\rangle-\langle u,w\rangle-\langle w,v\rangle+\langle u,v\rangle\right)\\ &= 2\|w\|^2+2\operatorname{Re}\left(-\langle u,w\rangle-\langle w,v\rangle+\langle u,v\rangle\right)\end{aligned}$$

2. Suppose that $u,v\in S$ are both closest to w:

$$\begin{aligned}\forall y\in S &: \|w-u\|\le\|w-y\|\\ \forall y\in S &: \|w-v\|\le\|w-y\|\end{aligned}$$

Take

$$y=\frac{u+v}{2}.$$

Then,

$$\begin{aligned}\|w-u\|^2 &\le \left\|w-\frac{1}{2}(u+v)\right\|^2\\ \|w-v\|^2 &\le \left\|w-\frac{1}{2}(u+v)\right\|^2\end{aligned}$$

thus

$$\frac{\|w-u\|^2+\|w-v\|^2}{2} \leq \left\|w-\frac{1}{2}(u+v)\right\|^2$$

and using part 1,

$$\frac{\|w-u\|^2+\|w-v\|^2}{2} \leq \left\|w-\frac{1}{2}(u+v)\right\|^2 = \frac{\|w-u\|^2+\|w-v\|^2}{2} - \frac{\|u-v\|^2}{4}$$

gives

$$\frac{\|u-v\|^2}{4} \leq 0$$

thus $\|u-v\|=0$ and $u=v$.

Problem 3. Let A be a nonsingular real $n \times n$ matrix. Prove that there exists a unique orthogonal matrix Q and a unique positive definite symmetric matrix B such that $A = QB$.

Solution: Since A is nonsingular, A^TA is positive definite. Let $B = \sqrt{A^TA}$, the unique positive definite symmetric matrix such that $B^2 = A^TA$. Consider $P = BA^{-1}$, i.e., $PA = B$. It is sufficient to show that P is orthogonal, since in this case, $Q = P^{-1} = P^T$ will be orthogonal and $A = QB$. We have

$$P^TP = (BA^{-1})^T(BA^{-1}) = (A^T)^{-1}B^TBA^{-1} = (A^T)^{-1}B^2A^{-1} = (A^T)^{-1}A^TAA^{-1} = I$$

Now we show uniqueness. Suppose we had a second factorization $A = Q_1B_1$. Then

$$B^2 = A^TA = B_1^TQ_1^TQ_1B_1 = B_1^TB_1 = B_1^2.$$

Since a positive definite matrix has a unique positive square root, it implies that $B = B_1$. On the other hand, since $Q_1 = AB_1^{-1}$ and $Q = AB^{-1}$ (B is invertible), $Q_1 = Q$.

Problem 4. We are in 2021, so let $A = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$. Define

$$\begin{array}{rcl} T : \mathcal{M}_2(\mathbb{R}) & \to & \mathcal{M}_2(\mathbb{R}) \\ B & \mapsto & AB - BA \end{array}$$

1. (5 points) Fix an ordered basis $\mathcal{B}$ of $\mathcal{M}_2(\mathbb{R})$ and compute the matrix $[T]_{\mathcal{B}}$ that represents T with respect to this basis.
2. (5 points) Give the eigenvalues of T.
3. (5 points) Compute a basis for each of the eigenspaces of T.
4. (5 points) Give the minimal and characteristic polynomials of T and the Jordan form for T. Say whether T is diagonalizable or not.

Solution:

1. We choose the "standard ordered basis"

$$\mathcal{B} = (E_{11}, E_{12}, E_{21}, E_{22})$$

where E_{ij} has a one in the (i, j) position and zero elsewhere.
Then routine computations show that

$$T(E_{11}) = AE_{11} - E_{11}A = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} = 0\ E_{11} + 0\ E_{12} + 2\ E_{21} + 0\ E_{22},$$
$$T(E_{12}) = AE_{12} - E_{12}A = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix} = -2\ E_{11} + 1\ E_{12} + 0\ E_{21} + 2\ E_{22},$$
$$T(E_{21}) = AE_{21} - E_{21}A = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = 0\ E_{11} + 0\ E_{12} - 1\ E_{21} + 0\ E_{22},$$
$$T(E_{22}) = AE_{22} - E_{22}A = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} = 0\ E_{11} + 0\ E_{12} - 2\ E_{21} + 0\ E_{22}.$$

The matrix $[T]_{\mathcal{B}}$ that represents T with respect to this basis is

$$T = \begin{pmatrix} 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & -2 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

2. We quickly note that since $T(E_{21}) = -E_{21}$, then E_{21} is eigenvector of eigenvalue -1. Also looking at matrix T, the four columns are clearly in a space of dimension 2, so the dimension of the nullspace of T is 2, so we will have an eigenspace for the eigenvalue 0 of dimension 2. And actually, thinking about $AB - BA$, it is clear

that if $B = I$, then $T(I) = 0$, so I is an eigenvector of eigenvalue 0, and it is also clear that, if $B = A$, then $T(A) = 0$, so A is an eigenvector of eigenvalue 0. Since A and I are linearly independent, we have found a basis for the eigenspace of eigenvalue 0. We know three eigenvalues out of four. And we also know three linearly independent eigenvectors. Now since the trace of T is 0 and the trace of T is the sum of the eigenvalues, we see that the fourth eigenvalue of T is 1. And so we are done. We found the four eigenvalues of T: 1, 0, 0, and -1.

With these observations, we also note that T will be diagonalizable since the sum of the dimensions of the eigenspaces is 4. The minimum polynomial and the characteristic polynomial will be the same and equal to $\lambda^2(\lambda+1)(\lambda-1)$. And the Jordan form of T will be a diagonal matrix with 1, 0, 0, and -1 on the diagonal. This answers part 4.

2. Here is another way to do part 2. To obtain the eigenvalues of T, we compute the characteristic polynomial

$$\begin{aligned}
\det(T-\lambda I) &= \begin{vmatrix} -\lambda & -2 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 2 & 0 & -1-\lambda & -2 \\ 0 & 2 & 0 & -\lambda \end{vmatrix} \\
&\quad \text{expand with respect } C_3 \\
&= (-1-\lambda)\begin{vmatrix} -\lambda & -2 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 2 & -\lambda \end{vmatrix} \\
&\quad \text{expand with respect } C_3 \\
&= (-1-\lambda)(-\lambda)\begin{vmatrix} -\lambda & -2 \\ 0 & 1-\lambda \end{vmatrix} \\
&\quad \text{expand with respect } C_1 \\
&= (-1-\lambda)(-\lambda)(-\lambda)(1-\lambda) \\
&= \lambda^2(\lambda+1)(\lambda-1)
\end{aligned}$$

We see that

$$\det(T-\lambda I) = \lambda^2(\lambda+1)(\lambda-1)$$

And so we find the eigenvalues of T are 1, 0, 0, and -1.

3. $\boxed{\lambda_1 = 1}$

We use the row reduction process on $T-I$ to find a basis for the nullspace of $T-I$, the eigenspace of T of eigenvalue 1.

$$T-I=\begin{pmatrix}-1&-2&0&0\\0&0&0&0\\2&0&-2&-2\\0&2&0&-1\end{pmatrix}\ \begin{matrix}L_1\leftarrow -L_1\\ \text{remove } L_2\\ L_3\leftarrow L_3/2\\ \rightsquigarrow\end{matrix}\ \begin{pmatrix}1&2&0&0\\1&0&-1&-1\\0&2&0&-1\end{pmatrix}\ \begin{matrix}L_2\leftarrow L_2-L_1\\ \rightsquigarrow\end{matrix}\ \begin{pmatrix}1&2&0&0\\0&-2&-1&-1\\0&2&0&-1\end{pmatrix}$$

$$\begin{matrix}L_3\leftarrow L_3+L_2\\ \rightsquigarrow\end{matrix}\ \begin{pmatrix}1&2&0&0\\0&-2&-1&-1\\0&0&-1&-2\end{pmatrix}\ \begin{matrix}L_3\leftarrow -L_3\\ \rightsquigarrow\end{matrix}\ \begin{pmatrix}1&2&0&0\\0&-2&-1&-1\\0&0&1&2\end{pmatrix}$$

$$\begin{matrix}L_2\leftarrow L_2+L_3\\ \rightsquigarrow\end{matrix}\ \begin{pmatrix}1&2&0&0\\0&-2&0&1\\0&0&1&2\end{pmatrix}\ \begin{matrix}L_1\leftarrow L_1+L_2\\ \rightsquigarrow\end{matrix}\ \begin{pmatrix}1&0&0&1\\0&-2&0&1\\0&0&1&2\end{pmatrix}$$

$$\begin{matrix}L_2\leftarrow L_2/(-2)\\ \rightsquigarrow\end{matrix}\ \begin{pmatrix}1&0&0&1\\0&1&0&(-1/2)\\0&0&1&2\end{pmatrix}$$

We have free variable is x_4, we set to $x_4=t$. We solve for x_1, x_2 and x_3 and we find:

$$\begin{cases}x_1=-t\\x_2=t/2\\x_3=-2t\\x_4=t\end{cases}\qquad x=t\begin{pmatrix}-1\\1/2\\-2\\1\end{pmatrix}\qquad v_1=\begin{pmatrix}-2\\1\\-4\\2\end{pmatrix}.$$

An eigenvector for the eigenvalue 1 is, for example,

$$B_1=\begin{pmatrix}-2&1\\-4&2\end{pmatrix}.$$

Indeed

$$T(B_1)=T(\begin{pmatrix}-2&1\\-4&2\end{pmatrix})=\begin{pmatrix}2&0\\2&1\end{pmatrix}\begin{pmatrix}-2&1\\-4&2\end{pmatrix}-\begin{pmatrix}-2&1\\-4&2\end{pmatrix}\begin{pmatrix}2&0\\2&1\end{pmatrix}=\begin{pmatrix}-2&1\\-4&2\end{pmatrix}=B_1.$$

$\boxed{\lambda_2=-1}$

We use the row reduction process on $T+I$ to find a basis for the nullspace of $T+I$, the eigenspace of T of eigenvalue -1.

$$T+I=\begin{pmatrix}1&-2&0&0\\0&2&0&0\\2&0&0&-2\\0&2&0&1\end{pmatrix}\ \begin{matrix}L_1\leftarrow L_1+L_2\\ L_4\leftarrow L_4-L_2\\ \rightsquigarrow\end{matrix}\ \begin{pmatrix}1&0&0&0\\0&2&0&0\\2&0&0&-2\\0&0&0&1\end{pmatrix}\ \begin{matrix}L_3\leftarrow L_3+2L_4\\ \rightsquigarrow\end{matrix}\ \begin{pmatrix}1&0&0&0\\0&2&0&0\\2&0&0&0\\0&0&0&1\end{pmatrix}$$

$$\begin{matrix}L_3\leftarrow L_3-2L_1\\ \rightsquigarrow\end{matrix}\ \begin{pmatrix}1&0&0&0\\0&2&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}\ \begin{matrix}L_2\leftarrow L_2/2\\ \text{remove } L_3\\ \rightsquigarrow\end{matrix}\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$$

We have free variable is x_3, we set to $x_3 = t$. We solve for x_1, x_2 and x_4 and we find:

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = t \\ x_4 = 0 \end{cases} \qquad x = t\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

An eigenvector for the eigenvalue -1 is, for example,

$$B_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Indeed

$$T(B_2) = T(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = -B_2.$$

$\boxed{\lambda_3 = 0}$

We use the row reduction process on T to find a basis for the nullspace of T, the eigenspace of T of eigenvalue 0.

$$T = \begin{pmatrix} 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & -2 \\ 0 & 2 & 0 & 0 \end{pmatrix} \begin{matrix} L_1 \leftarrow L_1 + 2L_2 \\ L_4 \leftarrow L_4 - 2L_2 \\ \rightsquigarrow \end{matrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \text{remove } L_1 \\ \text{remove } L_4 \\ \rightsquigarrow \end{matrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & -2 \end{pmatrix}$$

We have free variable is x_3 and x_4, we set to $x_3 = s$ and $x_4 = t$. We solve for x_1 and x_2 and we find:

$$\begin{cases} x_1 = s/2 + t \\ x_2 = 0 \\ x_3 = s \\ x_4 = t \end{cases} \qquad x = s\begin{pmatrix} 1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} \text{ and } v_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

A basis eigenvectors for the eigenspace associated with the eigenvalue 0 is, for example,

$$B_3 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \text{ and } B_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Indeed

$$T(B_3) = T(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

$$T(B_4) = T(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

Note that we already had identified another basis for the eigenspace of eigenvalue 0. We observed that, if $B = I$, then $T(I) = AI - IA = 0$, so I is an eigenvector of eigenvalue 0. And we observed that, if $B = A$, then $T(A) = AA - AA = 0$, so A is an eigenvector of eigenvalue 0. Since A and I are linearly independent, another basis for the eigenspace of eigenvalue 0 is (I, A).

4. T is diagonalizable since the sum of the dimensions of the eigenspaces is 4. The characteristic polynomial is equal to $\lambda^2(\lambda+1)(\lambda-1)$. Because T is diagonalizable, the minimum polynomial is equal to $\lambda(\lambda+1)(\lambda-1)$. The Jordan form of T will be a diagonal matrix with 1, 0, 0, and -1 on the diagonal.

Part II. Work **two** of problems 5 through 8.

Problem 5. Define $\mathbb{R}^{n\times n}$ to be the space of all real n-by-n matrices, suppose $S \in \mathbb{R}^{n\times n}$, and define the linear mapping

$$\mathcal{T} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}, \quad \mathcal{T} : P \mapsto PS + SP$$

1. (10 points) Prove that if λ is eigenvalue of S, u is the corresponding eigenvector, and $u \in \operatorname{null}(\mathcal{T}P)$, then Pu is also an eigenvector of S, with eigenvalue $-\lambda$.
2. (10 points) Prove that if S is symmetric positive definite, then the mapping $\mathcal{T}$ is injective.

Solution:

1. Suppose $u \in \operatorname{null}(\mathcal{T}P)$ and $Su = \lambda u$. Then,

   $$0 = (PS + SP)\,u = P(Su) + S(Pu) = P(\lambda u) + S(Pu) = \lambda Pu + S(Pu),$$

   thus

   $$S(Pu) = -\lambda Pu.$$

2. A linear mapping is injective if and only if its kernel is $\{0\}$. Therefore, we need to show that the only solution $S \in \mathbb{R}^{n\times n}$ of

   $$PS + SP = 0 \tag{2}$$

   is $P = 0$. Since S is symmetric, there exists basis $u_1, \ldots, u_n$ of $\mathbb{R}^n$ consisting of eigenvectors of S,

   $$Su_k = \lambda_k u_k,$$

   and since S is positive definite, all eigenvalues $\lambda_k > 0$. Then, for each k, if $Pu_k \neq 0$, then Pu_k is eigenvector of S with eigenvalue $-\lambda_k < 0$ from the first part of the problem, which is a contradiction with S being positive definite. Thus, $Pu_k = 0$. Since the linear operator $u \mapsto Pu$ is zero on all vectors of a basis, it is zero operator, and $P = 0$.

Problem 6.

Let V be a vector space of dimension n over a field F. For any nilpotent operator T on V, define the smallest integer p such that $T^p = 0$ as the index of nilpotency of T.

1. (6 points) Suppose that N is nilpotent of index p. If $\boldsymbol{v} \in V$ is such that $N^{p-1}(\boldsymbol{v}) \neq 0$, prove that

$$\{\boldsymbol{v}, N(\boldsymbol{v}), \ldots, N^{p-1}(\boldsymbol{v})\}$$

is linearly independent.

2. (7 points) Show that N is nilpotent of index n if and only if there is an ordered basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ of V such that the matrix of N with respect to the basis is of the form

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

3. (7 points) Show that an $n \times n$ matrix M over F is such that $M^n = 0$ and $M^{n-1} \neq 0$ if and only if M is similar to a matrix of the above form.

Solution:

1. If $\boldsymbol{v} \neq \boldsymbol{0}$ is such that $N^{p-1}(\boldsymbol{v}) \neq \boldsymbol{0}$, then $\boldsymbol{v}, N(\boldsymbol{v}), \ldots, N^{p-1}(\boldsymbol{v})$ are all non-zero. Suppose that

$$a_0\boldsymbol{v} + a_1 N(\boldsymbol{v}) + \cdots + a_{p-1}N^{p-1}(\boldsymbol{v}) = \boldsymbol{0}$$

Applying N^{p-1} to both sides, we obtain $a_0 N^{p-1}\boldsymbol{v} = 0$, which implies $a_0 = 0$. Thus we have

$$a_1 N(\boldsymbol{v}) + \cdots + a_{p-1}N^{p-1}(\boldsymbol{v}) = \boldsymbol{0}$$

Applying N^{p-2} to both sides, similarly we can obtain $a_1 = 0$. Following the pattern, we can show all the coefficients $a_j, j = 1, \ldots, p-1$ are 0. So the set is linearly independent.

2. Clearly $\boldsymbol{v}, N(\boldsymbol{v}), \ldots, N^{p-1}(\boldsymbol{v})$ is a basis of V. From the following equations:

$$\begin{aligned} N(\boldsymbol{v}) &= 0\boldsymbol{v} + 1N(\boldsymbol{v}) + 0N^2(\boldsymbol{v}) + \cdots + 0N^{n-1}(\boldsymbol{v}) \\ N^2(\boldsymbol{v}) &= 0\boldsymbol{v} + 0N(\boldsymbol{v}) + 1N^2(\boldsymbol{v}) + \cdots + 0N^{n-1}(\boldsymbol{v}) \\ &\vdots \\ N^{p-1}(\boldsymbol{v}) &= 0\boldsymbol{v} + 0N(\boldsymbol{v}) + 0N^2(\boldsymbol{v}) + \cdots + 1N^{n-1}(\boldsymbol{v}) \\ N^n(\boldsymbol{v}) &= 0\boldsymbol{v} + 0N(\boldsymbol{v}) + 0N^2(\boldsymbol{v}) + \cdots + 0N^{n-1}(\boldsymbol{v}) \end{aligned}$$

we obtain the matrix of N relative to the above ordered basis is:

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Conversely, suppose that there is an ordered basis with respect to which the matrix A of N is of the above form. It is easy to check that $A^n = 0$ and $A^{n-1} \neq 0$. Consequently N is nilpotent of index n.

3. If M is similar to a matrix of the above form, denoted by A, then $M = P^{-1}AP$, where P is invertible. Then

$$M^{n-1} = P^{-1}A^{n-1}P \neq 0, \text{ and } M^n = P^{-1}A^nP = P^{-1}0P = 0$$

Conversely, if M is an $n \times n$ matrix over F such that $M^n = 0$ and $M^{n-1} \neq 0$, then M is an nilpotent matrix of index n. Let N be the nilpotent operator of index n whose matrix is M with respect to a certain basis, then based on the second part, A matrix of N has the form of A under (potentially) another basis. So M and A must be similar.

Problem 7. Let S, T be two normal operators in the complex finite dimensional inner product space V such that $ST = TS$. Prove that there is a basis for V consisting of vectors that are eigenvectors of both S and T.

Solution: Let $n = \dim V$, and $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of S. Then by the spectral theorem, there exist eigenvectors of S that form an orthonormal basis of V, which means:

$$E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k} = V.$$

For any $i \in \{1, 2, \ldots, k\}$ and $\boldsymbol{v} \in E_{\lambda_i}$, we have $S\boldsymbol{v} = \lambda_i \boldsymbol{v}$. Then

$$TS\boldsymbol{v} = T(\lambda_i \boldsymbol{v}) \Rightarrow ST\boldsymbol{v} = \lambda_i T\boldsymbol{v} \Rightarrow S(T\boldsymbol{v}) = \lambda_i (T\boldsymbol{v}).$$

So $T\boldsymbol{v} \in E_{\lambda_i}$, and hence E_{λ_i} is T-invariant. Then $T|_{E_{\lambda_i}}$ is a normal operator on E_{λ_i}. By the Spectral Theorem, there exists an orthonormal basis $\boldsymbol{u}_1^i, \ldots, \boldsymbol{u}_{n_i}^i$ for E_{λ_i} consisting of eigenvectors of T. Then

$$\bigcup_{i=1}^{k} \bigcup_{j=1}^{n_i} \{\boldsymbol{u}_j^i\}$$

is a basis of V consisting of eigenvectors of both S and T.

Problem 8. Let $\mathcal{M}_2(\mathbb{C})$ be the set of 2-by-2 matrices with coefficients in $\mathbb{C}$, and $A \in \mathcal{M}_2(\mathbb{C})$. Define

$$\mathcal{S} = \left\{ N \in \mathcal{M}_2(\mathbb{C}) | B = \begin{pmatrix} A & N \\ 0 & A \end{pmatrix} \text{ is diagonalizable} \right\}.$$

1. (10 points) Suppose that

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \lambda_1 \neq \lambda_2,$$

and prove that $\mathcal{S}$ is a 2-dimensional subspace of $\mathcal{M}_2(\mathbb{C})$. Hint: First consider $B \in \mathcal{S}$ and find the conditions which N must satisfy for the eigenspaces of B to have the required dimensions such that B is diagonalizable.

2. (10 points) Prove that for any $A \in \mathcal{M}_2(\mathbb{C})$ with two distinct eigenvalues λ_1 and λ_2, $\mathcal{S}$ is a 2-dimensional subspace of $\mathcal{M}_2(\mathbb{C})$. Hint: Transform this to the previous case.

Solution:

1. The case where A is a diagonal matrix with two distinct diagonal entries λ_1 and λ_2

   Now let us consider
$$N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
   such that
$$B = \begin{pmatrix} A & N \\ 0 & A \end{pmatrix}$$
   is diagonalizable. We have that
$$B = \begin{pmatrix} \lambda_1 & 0 & a & b \\ 0 & \lambda_2 & c & d \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}.$$
   Clearly B has eigenvalues λ_1 and λ_2, both of algebraic multiplicities 2. Since B is diagonalizable, the geometric multiplicities of λ_1 and λ_2 is 2. In other words, since B is diagonalizable,
$$\dim \text{Null}(B - \lambda_1 I) = 2 \quad \text{and} \quad \dim \text{Null}(B - \lambda_2 I) = 2.$$
   Let us do row reduction on $B - \lambda_1 I$. (Row reduction uses elementary row operations, which conserve the nullspace.)
$$B - \lambda_1 I = \begin{pmatrix} 0 & 0 & a & b \\ 0 & \lambda_2 - \lambda_1 & c & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 - \lambda_1 \end{pmatrix}.$$
   Now we can remove the third row (all zero), and, since $\lambda_2 - \lambda_1$ is not zero, we can divide the second and fourth row by $\lambda_2 - \lambda_1$. If we call $c' = c/(\lambda_2 - \lambda_1)$ and $d' = c/(\lambda_2 - \lambda_1)$, we get
$$B - \lambda_1 I \sim \begin{pmatrix} 0 & 0 & a & b \\ 0 & 1 & c' & d' \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
   Let us permute row 1 and 2.
$$B - \lambda_1 I \sim \begin{pmatrix} 0 & 1 & c' & d' \\ 0 & 0 & a & b \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let us introduce a zero in position (4,1) by doing $L_1 \leftarrow L_1 - d'L_3$ and introduce a zero in position (4,2) by doing $L_2 \leftarrow L_2 - bL_3$. We get

$$B - \lambda_1 I \sim \begin{pmatrix} 0 & 1 & c' & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

So now, for sake of contradiction, assume that a is not zero, then we have one free variable and three leading variables, so the dimension of the nullspace of $B - \lambda_1 I$ is 1, and this is a contradiction since, since B is diagonalizable, $\dim \operatorname{Null}(B - \lambda_1 I) = 2$. This means that the assumption a is not zero is false and so we must have

$$a = 0.$$

And we can check that if a is zero, indeed, $\dim \operatorname{Null}(B - \lambda_1 I) = 2$ and all is well.

Similarly, by doing row reduction on $(B - \lambda_2 I)$ and using the fact that $\dim \operatorname{Null}(B - \lambda_2 I) = 2$, we see that we must have

$$d = 0.$$

This reasoning shows that for B to be diagonalizable, we must have N of the form

$$N = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

Reciprocally, if N is of the form

$$N = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix},$$

then

$$\dim \operatorname{Null}(B - \lambda_1 I) = 2 \quad \text{and} \quad \dim \operatorname{Null}(B - \lambda_2 I) = 2,$$

and so B is diagonalizable.

Therefore,

$$\begin{aligned} \mathcal{S} &= \left\{ N \in \mathcal{M}_2(\mathbb{C}) \text{ such that } B = \begin{pmatrix} A & N \\ 0 & A \end{pmatrix} \text{ is diagonalizable} \right\} \\ &= \left\{ N = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}, \text{ for all } b \text{ and all } c \right\} \end{aligned}$$

In this second form, the subset $\mathcal{S}$ is clearly a subspace of $\mathcal{M}_2(\mathbb{C})$ of dimension 2.

(A basis for this subspace is, for example, $(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})$.)

2. The case where A is a general 2-by-2 matrix with two distinct eigenvalues λ_1 and λ_2

Let v_1 be an eigenvector associated with λ_1. Let v_2 be an eigenvector associated with λ_2. Let V, the 2-by-2 matrix, $V = (v_1, v_2)$. Then we have

$$A = V \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} V^{-1}.$$

(Note that the matrix V is invertible because the set (v_1, v_2) is linearly independent because these are eigenvectors associated with distinct eigenvalues.)

Now let us define W as

$$W = \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix},$$

let us define a, b, c and d as

$$V^{-1}NV = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and let us look at $W^{-1}BW$.

We have

$$\begin{aligned} W^{-1}BW &= \begin{pmatrix} V^{-1} & 0 \\ 0 & V^{-1} \end{pmatrix} \begin{pmatrix} A & N \\ 0 & A \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} \\ &= \begin{pmatrix} V^{-1}AV & V^{-1}NV \\ 0 & V^{-1}AV \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 & 0 & a & b \\ 0 & \lambda_2 & c & d \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \end{aligned}$$

Which looks exactly like "the case when A is a diagonal matrix with two distinct diagonal entries λ_1 and λ_2".

Now since B is diagonalizable, and W is invertible, we have that $W^{-1}BW$ is diagonalizable. By the "case 1", it must therefore be that $a = 0$ and $d = 0$. Therefore we have that

$$V^{-1}NV = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$$

and so

$$N = V \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} V^{-1}.$$

Reciprocally if N is of the form

$$N = V \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} V^{-1}$$

we see by a similar reasonning, and using case 1, that B is diagonalizable

Therefore,

$$\begin{aligned}\mathcal{S} &= \left\{ N \in \mathcal{M}_2(\mathbb{C}) \text{ such that } B \begin{pmatrix} A & N \\ 0 & A \end{pmatrix} \text{ is diagonalizable} \right\} \\ &= \left\{ N = V \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} V^{-1}, \text{ for all } b \text{ and all } c \right\}\end{aligned}$$

In this second form, it is easy to prove that the subset $\mathcal{S}$ is a subspace of $\mathcal{M}_2(\mathbb{C})$ of dimension 2. A basis for this subspace is, for example, $(V \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} V^{-1}, V \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} V^{-1})$.
