Analysis Prelim—January 22, 2021
Solutions

Section 1

1. For two subsets A and B of metric space X consider set S of all points $x \in X$ such that x is a limit point for both sets A and B and not an interior point for either A or B. Prove that S is closed. You can use well known theorems in your proof if you carefully state them.

   Solution: From the hint, we expect you to know (and use the fact) that the set of limit points of a set is closed (e.g., Rudin, exercise 2.6) and that the interior of a set is open (e.g., Rudin. exercise 2.9). We also assume you know that if O is open and C is closed, then $C - O$ is closed.

   Let A' and B' be the limit points of A and B. Then each is closed, so $A' \cap B'$ is closed. Likewise, $A^o \cup B^o$ is open. Thus, the set in question, $S = (A' \cap B') - (A^o \cup B^o)$, is closed.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous, and let $c_n \searrow 0$. Define $f_n(x) = f(x + c_n),\ x \in \mathbb{R}$.

   (a) (15 points) Prove that $f_n \to f$ uniformly.

   (b) (5 points) If f is continuous (but not uniformly continuous) is the result still true? Prove or find a counterexample.

   Solution: (a) Choose $\epsilon > 0$ and then $\delta > 0$ so that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$. Choose N so that $n > N \Rightarrow |c_n| < \delta$. Then for $n > N$ and $x \in \Re$,

   $$|f_n(x) - f(x)| = |f(x + c_n) - f(x)| < \epsilon$$

   so $f_n \to f$ uniformly.

   (b) Let $f(x) = e^x$ and suppose that every $c_n > 0$. Then $\forall n, \exists x_n \in \Re$ such that

   $$|f_n(x_n) - f(x_n)| = e^{x_n}|e^{c_n} - 1| > 1$$

   so the convergence is not uniform.

3. Prove that $\sum_{k=1}^{\infty} \frac{1}{k} \cos\left(\frac{2\pi}{3}k\right)$ converges.

   Solution. The cosine terms in the sum form a repeating pattern, $\{-\frac{1}{2}, -\frac{1}{2}, 1, \ldots\}$, so the sum is

   $$-\left[\frac{1}{2}(1 + \frac{1}{2})\right] + \frac{1}{3} - \left[\frac{1}{2}(\frac{1}{4} + \frac{1}{5})\right] + \frac{1}{6} - \cdots,$$

   which converges by the alternating series test once we note that $\frac{1}{k} > \frac{1}{2}(\frac{1}{k+1} + \frac{1}{k+2}) > \frac{1}{k+3}$.

4. Let $f_n(x) = nxe^{-nx^2},\ n = 1, 2, \ldots$.

   (a) (5 points) Prove that $f_n(x) \to 0$ pointwise on $[0, 1]$.

   (b) (5 points) Find $\lim_{n\to\infty} \int_0^1 f_n(x)dx$. Hint: The answer is not 0.

   (c) (5 points) Use (a) and (b) to prove that the convergence $f_n(x) \to 0$ on $[0, 1]$ is not uniform.

   (d) (5 points) Prove the convergence $f_n(x) \to 0$ on $[0, 1]$ is not uniform directly from the definition of uniform convergence.

Solution: (a) For every n, $f_n(0) = 0$. For $x \in (0, 1]$, note that

$$e^{nx^2} = \sum_{i=0}^{\infty} \frac{(nx^2)^i}{i!} > \frac{n^2x^4}{2},$$

so for every $x \in (0, 1]$

$$f_n(x) = \frac{nx}{e^{nx^2}} < \frac{2}{nx^3} \to 0.$$

(b) $\int_0^1 f_n(x)dx = (1 - e^{-n})/2 \to 1/2$.

(c) By Rudin, Theorem 7.16, if $f_n(x) \to 0$ uniformly then

$$1/2 = \lim \int_0^1 f_n(x)dx = \int_0^1 \lim f_n(x)dx = 0$$

which is a contradiction, so the convergence cannot be uniform.

(d) Let $x_n = 1/\sqrt{n}$. Then $f_n(x_n) = \sqrt{n}e^{-1} \not\to 0$, so the convergence is not uniform.

Section 2

5. Let (X, d) be a metric space and $A \subset X$ be compact and non-empty. Let $f : A \to A$ be a continuous function such that for all $x, y \in A$, $d(f(x), f(y)) \geq d(x, y)$ (f is a non-contracting function). Prove that
(a) (5 points) f is one-to-one
(b) (5 points) f^{-1} is continuous
(c) (10 points) $f(A) = A$. Hint: Suppose $y_0 \in A - f(A)$. Consider the sequence defined by $y_n = f(y_{n-1})$.

Solution: (a) Suppose $f(x) = f(y)$ for some $x \neq y$. Then $0 = d(f(x), f(y)) \geq d(x, y) \geq 0$ which means $d(x, y) = 0$, so $x = y$.

(b) Suppose $y_n \to y$ in $f(A)$. Then $d(y_n, y) \to 0$. But $d(y_n, y) \geq d(f^{-1}(y_n), f^{-1}(y))$, so $d(f^{-1}(y_n), f^{-1}(y)) \to 0$, which means $f^{-1}(y_n) \to f^{-1}(y)$. Since (y_n) was an arbitrary convergent sequence, f^{-1} is continuous.

(c) Suppose $\exists x \in A$, $x \notin f(A)$. Since f is continuous and A is compact, $f(A)$ is also compact. So there is $B_\epsilon(x) \subset A^c$. Let $y_0 = x \notin f(A)$, $y_1 = f(x) \in f(A)$, and for $n > 1$ define $y_n = f(y_{n-1}) \in f(A)$. Note that for $n > m$,

$$d(y_n, y_m) = d(f(y_{n-1}), f(y_{m-1})) \geq d(y_{n-1}, y_{m-1}) \cdots \geq d(x, y_{n-m}) > \epsilon.$$

This means the elements of the sequence (y_n) are at a distance from each other of at least ϵ, so (y_n) has no convergent (Cauchy) subsequence. But, since $f(A)$ is compact, this is a contradiction.

6. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$. Prove that f is uniformly continuous if and only if $\forall (x_n), (z_n) \subset X$, $d_X(x_n, z_n) \to 0 \Rightarrow d_Y(f(x_n), f(z_n)) \to 0$.

Solution: ($\Rightarrow$): f is uniformly continuous. Choose $\epsilon > 0$. Then $\exists \delta > 0$ such that $d_X(x, z) < \delta \Rightarrow d_Y(f(x), f(z)) < \epsilon$. Suppose $d(x_n, z_n) \to 0$. Choose N so that $n > N \Rightarrow d_X(x_n, z_n) < \delta$. Then $n > N \Rightarrow d_Y(f(x_n), f(z_n)) < \epsilon$.

($\Leftarrow$): If f is not uniformly continuous then for some $\epsilon > 0$, $\exists x_n, z_n \in X$ such that $d_X(x_n, z_n) < 1/n$ but $d_Y(f(x_n), f(z_n)) > \epsilon$. Thus, $d_X(x_n, z_n) \to 0$ but $d_Y(f(x_n), f(z_n)) \not\to 0$.

7. Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove these two standard theorems. If you are not sure if you are allowed to use some other theorem in the proof, ask your friendly proctor.

(a) (10 points) f is Riemann integrable on $[a, b]$.

(b) (10 points) f is bounded on $[a, b]$.

Solution: These are both standard results, e.g., (a) is Rudin, Theorem 6.18, and (b) is Rudin, Theorem 4.15.

8. Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of functions that converges pointwise to f.

   (a) (10 points) Prove or find a counterexample. If each f_n is continuous then f is integrable.

   (b) (10 points) Is $f_n(x) = x^{\frac{1}{n}} \sin^{2n+1}(\frac{1}{x})$ a counterexample for (a)? Explain carefully.

Solution: (a) Here's a counterexample:

$$f_n(x) = \frac{x}{x^2 + \frac{1}{n}} \to \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{if } 0 < x \leq 1 \end{cases}$$

(b) This example is not a counterexample since,

$$f_n(x) \to f(x) = \begin{cases} +1 & \text{if } x = \frac{1}{\frac{\pi}{2} + 2\pi k} \\ -1 & \text{if } x = \frac{1}{\frac{3\pi}{2} + 2\pi k} \\ 0 & \text{otherwise} \end{cases}$$

But f is Riemann integrable since it is bounded, and for any $\epsilon > 0$, $f(x) = 0$ at all but a finite number of points outside the interval $[0, \epsilon]$.