Analysis Prelim—January 22, 2021

- Be sure to show all your work that is relevant for each problem, but do not turn in scratch work. Rewrite your solutions neatly if they are hard to read or not well organized.
- Start a new sheet of paper for every problem, and write your name and the problem number on every sheet. If you are using a tablet computer to write your solutions, make sure each problem is separate and easy to find.
- If you use a statement from Rudin, Pugh, or class, state it. If you are unsure if a statement must be proved or may merely be stated, ask your friendly proctor.
- Each problem is worth a total of 20 points. The weights for each part on multi-step problems are marked.
- The exam has two sections. Do all four problems from the first section, and two of the four problems from the second section. Do not turn in more than two solutions for the second part. If you do, we will count the two lowest scores. A perfect score on the exam is 120 points.
- This is a 4 hour exam. When the proctor says the time is up, stop working, create a single pdf file of your solutions, and email it to burt.simon@ucdenver.edu.
- Good luck!

Name:

1	2	3	4	5	6	7	8	$\sum$

Section 1: do all four problems

1. For two subsets A and B of metric space X consider set S of all points $x \in X$ such that x is a limit point for both sets A and B and not an interior point for either A or B. Prove that S is closed. You can use well known theorems in your proof if you carefully state them.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous, and let $c_n \searrow 0$. Define $f_n(x) = f(x + c_n),\ x \in \mathbb{R}$.

   (a) (15 points) Prove that $f_n \to f$ uniformly.

   (b) (5 points) If f is continuous (but not uniformly continuous) is the result still true? Prove or find a counterexample.

3. Prove that $\sum_{k=1}^{\infty} \frac{1}{k} \cos\left(\frac{2\pi}{3} k\right)$ converges.

4. Let $f_n(x) = nxe^{-nx^2},\ n = 1, 2, \ldots$.

   (a) (5 points) Prove that $f_n(x) \to 0$ pointwise on $[0, 1]$.

   (b) (5 points) Find $\lim_{n\to\infty} \int_0^1 f_n(x)dx$. Hint: The answer is not 0.

   (c) (5 points) Use (a) and (b) to prove that the convergence $f_n(x) \to 0$ on $[0, 1]$ is not uniform.

   (d) (5 points) Prove the convergence $f_n(x) \to 0$ on $[0, 1]$ is not uniform directly from the definition of uniform convergence.

Section 2: do two of the following four problems

5. Let (X, d) be a metric space and $A \subset X$ be compact and non-empty. Let $f : A \to A$ be a continuous function such that for all $x, y \in A$, $d(f(x), f(y)) \geq d(x, y)$ (f is a non-contracting function). Prove that
   (a) (5 points) f is one-to-one
   (b) (5 points) f^{-1} is continuous
   (c) (10 points) $f(A) = A$. Hint: Suppose $y_0 \in A - f(A)$. Consider the sequence defined by $y_n = f(y_{n-1})$.

6. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$. Prove that f is uniformly continuous if and only if $\forall (x_n), (z_n) \subset X,\ d_X(x_n, z_n) \to 0 \Rightarrow d_Y(f(x_n), f(z_n)) \to 0$.

7. Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove these two standard theorems. If you are not sure if you are allowed to use some other theorem in the proof, ask your friendly proctor.

   (a) (10 points) f is Riemann integrable on $[a, b]$.

   (b) (10 points) f is bounded on $[a, b]$.

8. Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of functions that converges pointwise to f.

   (a) (10 points) Prove or find a counterexample. If each f_n is continuous then f is integrable.

   (b) (10 points) Is $f_n(x) = x^{\frac{1}{n}} \sin^{2n+1}(\frac{1}{x})$ a counterexample for (a)? Explain carefully.