Analysis Prelim—August 3, 2020

- Be sure to show all your work that is relevant for each problem, but do not turn in scratch work. Rewrite your solutions neatly if they are hard to read or not well organized.
- Start a new sheet of paper for every problem, and write your name and the problem number on every sheet. If you are using a tablet computer to write your solutions, make sure each problem is separate and easy to find.
- If you use a statement from Rudin, Pugh, or class, state it. If you are unsure if a statement must be proved or may merely be stated, ask your friendly proctor.
- Each problem is worth a total of 20 points. The weights for each part on multi-step problems are marked.
- The exam has two sections. Do all four problems from the first section, and two of the four problems from the second section. Do not turn in more than two solutions for the second part. If you do, we will count the two lowest scores. A perfect score on the exam is 120 points.
- This is a 4 hour exam. When the proctor says the time is up, stop working, create a single pdf file of your solutions, and email it to burt.simon@ucdenver.edu.
- Good luck!

Name:

1	2	3	4	5	6	7	8	$\sum$

Section 1: do all four problems

1. (20 points) Let B be a set in a metric space (X, d). Let B' be the set of all limit points of B. State the definition of a limit point and prove that B' is closed.

2. (20 points) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be uniformly continuous. Let (x_n) be a Cauchy sequence in X. Prove that $(f(x_n))$ is a Cauchy sequence in Y.

3. (20 points) Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable at $x = 0$ and discontinuous at every other $x \in \mathbb{R}$. Prove that your example works.

4. (20 points) For all $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be a function with at most one discontinuity. Further assume that $f_n \to f$ uniformly for some function $f : \mathbb{R} \to \mathbb{R}$. Prove that f has at most one discontinuity.

Section 2: do two of the following four problems

5. Let $f : [0, 1] \to \mathbb{R}$ have the following property. For every $x \in [0, 1]$ and $\varepsilon > 0$, there exists $\delta > 0$, such that for every $y \in [0, 1]$ with $|x - y| < \delta$, we have $f(y) < f(x) + \varepsilon$.

   (a) (10 points) Prove that f is bounded above, and

   (b) (10 points) f attains its maximum (i.e., there exists $z \in [0, 1]$ such that $f(x) \leq f(z)$ for all $x \in [0, 1]$).

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with

$$\lim_{x\to\infty} \left(f(x) + \int_0^x f(t)\ dt \right) = 0.$$

   (a) (5 points) Suppose $\lim_{x\to\infty} f(x) \neq 0$. Prove $\exists \epsilon > 0$ such that for any $z > 0$, $\exists\ x > z$ and $y > x$ such that $|f(x)| < \epsilon/2$ and $|f(y)| > \epsilon$.

   (b) (15 points) Use (a) to prove that $\lim_{x\to\infty} f(x) = 0$.

7. (20 points) Let M be a metric space. Prove that if every nested decreasing sequence (X_n) of closed nonempty subsets $M \supseteq X_1 \supseteq X_2 \supseteq \dots$ has $\bigcap X_n \neq \emptyset$, then M is compact. Hint: You can use without proof that sequentially compact metric spaces are compact.

8. Let M be a compact metric space, let $f_n : M \to \mathbb{R}$ be a sequence of continuous functions, such that for all $x \in M$, $(f_n(x))$ is monotonically decreasing with limit 0. Prove that f_n converges uniformly to the zero function. Here is one path to a proof, but you can try another, e.g., the result from problem 7.

   (a) (5 points) Argue that each f_n attains its maximum at some $x_n \in M$.

   (b) (5 points) Prove that $(f_n(x_n))$ is a monotonically decreasing sequence.

   (c) (10 points) Argue from (a) and (b) that $f_n \to 0$ uniformly.