University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam Solutions
August 10, 2020

Name: ____

Exam Rules:

- This exam is being administered remotely using Zoom. During the exam, you must be logged in to the assigned Zoom meeting with your your camera on, and you must be visible in the camera. Your microphone can be muted, but please leave Zoom audio ON so that the proctor can speak to you if needed.
- If you have any questions during the exam, please contact the exam proctor using the zoom chat, or call the proctor at the number given below.

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- If you would like to work with a paper copy of the exam, please print it as soon as you receive it and inform the proctor that you are doing so before the exam begins. If the printer is in another room, let the proctor know.
- You may read the exam as soon as you receive it, but you may not start writing (even your name) until authorized to start writing.
- This is a closed book exam. You may not use any external aids during the exam, such as:
  - communicating with anyone other than the exam proctor (through text messages or emails, for example);
  - consulting the internet, textbooks, notes, solutions of previous exams, etc;
  - using calculators or mathematical software.
- You may use a tablet PC (such as an iPad or Microsoft Surface) to write your solutions. Alternatively, you can write your solutions on paper.
- Please begin each problem on a new page, and write the problem number and page number at the top of each page. (For example, 6-1, 6-2, 6-3 for pages 1, 2 and 3 of problem 6). If you are writing on paper, write only on one side of the paper.
- The exam will end 4 hours after it begins. At the conclusion of the exam, please email a copy of your solutions to the exam proctor. Do not leave until the proctor acknowledges that your solutions have been successfully received.

- Your solutions need to be in a single .pdf file with the pages in the correct order. The .pdf file needs to be of good enough quality for easy grading.
- If you cannot create a good quality .pdf file quickly, you may instead submit an imperfect scan, or even pictures of your exam, and then take more time to prepare and submit a good quality .pdf version. We will grade the better version but use the first submission to check that nothing was added or changed between versions.
- Do not submit your scratch work.

- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded. Each problem is worth 20 points.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Notation: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{n,n}$ are the vector spaces of n-tuples and $n \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- If you are confused or stuck on a problem, either ask a question or move on to another problem.

Problem	Points	Score		Problem	Points	Score
1.	20			5.	20	
2.	20			6.	20	
3.	20			7.	20	
4.	20			8.	20	
				Total	120	

Applied Linear Algebra Preliminary Exam Committee:
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Part I. Work **all** of problems 1 through 4.

Problem 1.

(a) Let $A = \begin{bmatrix} 2 & -1 & 3 & -1 & 1 & -5 \\ -1 & 4 & 2 & 11 & -3 & 10 \\ 1 & 1 & 3 & 4 & 2 & -7 \end{bmatrix}$. Find bases for the null and column spaces of A.

(b) Let S and T be subspaces of $\mathbb{R}^n$.

(i) Show that there exist matrices A and B such that $S = \text{null } A$ and $T = \text{null } B$. Given bases for S and T, describe briefly how such matrices A and B could be found.

(ii) Find a matrix C such that $S \bigcap T = \text{null } C$.

(iii) Describe briefly how to find a basis for $S \bigcap T$.

Solution:

(a) The reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & -1 \\ 0 & 1 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{bmatrix}.$$

From this, we see that every solution to $Ax = 0$ satisfies

$$\begin{aligned} x_1 &= -2x_3 - x_4 + x_6 \\ x_2 &= -x_3 - 3x_4 \\ x_5 &= 3x_6, \end{aligned}$$

which gives the general solution

$$x = x_3 \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}.$$

Thus, a basis for null A is

$$\left\{ \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$$

Any set of 3 linearly independent columns of A form a basis for the column space. Since the reduced row echelon form of A has a pivot in columns 1,2 and 5, it follows that those 3 columns are linearly independent. So a basis for Col A is

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \right\}$$

(b) (i) Let P be a $p \times n$ matrix whose rows form a basis of S. By the rank-nullity theorem, $\dim \text{null } P = n - p$. Using the method from part 1, we can find a basis $\{a_1, \ldots, a_{n-p}\}$ for null P. Define

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_{n-p}^T \end{bmatrix}.$$

Since the rows of A form a basis for null P, we have $AP^T = 0$. Thus, $S =$ Row $P \subset$ null A. By the rank-nullity theorem, $\dim \text{null } A = n - (n - p) = p = \dim S$. Thus, $S =$ null A.

A similar argument can be used to construct a matrix B satisfying $T =$ null B.

(ii) Define

$$C = \begin{bmatrix} A \\ B \end{bmatrix}.$$

Then,

$$x \in \text{null } C \iff x \in \text{null } A \text{ and } x \in \text{null } C \iff x \in S \text{ and } x \in T \iff x \in S \bigcap T.$$

Thus, null $C = S \bigcap T$.

(iii) A basis for $S \bigcap T$ can be found by solving the system $Cx = 0$ to obtain a general solution in parametric vector form: $x = \alpha_1 v_1 + \ldots \alpha_k v_k$. The set of vectors $\{v_1, \ldots, v_k\}$ is a basis for $S \bigcap T$.

Problem 2.

Let V be a finite-dimensional real inner product space. Let $T \in \mathcal{L}(V)$ and let U be any T-invariant subspace of V. Prove or give a counterexample for each of the following statements:

(a) $U^\perp$ is T^*-invariant.

(b) $U^\perp$ is T-invariant.

Solution:

(a) Proof: Let $w \in U^\perp$. For any $u \in U$, $T(u) \in U$, since U is T-invariant. Thus, $0 = \langle w, T(u)\rangle = \langle T^*(w), u\rangle$, so $T^*(w) \in U^\perp$. It follows that $U^\perp$ is T^*-invariant.

(b) Counterexample: Let $V = \mathbb{R}^2$ with the usual inner product. Define $T \in \mathcal{L}(V)$ by $T(a, b) = (b, 0)$ and let $U = \{(a, 0)|a \in \mathbb{R}\}$. Then $U^\perp = \{(0, b)|b \in \mathbb{R}\}$. Observe that $T(a, 0) = (0, 0) \in U$, so U is T-invariant. But, for $b \neq 0$, $T(0, b) = (b, 0) \notin U^\perp$, so $U^\perp$ is not T-invariant.

Problem 3.

For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$, let $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ be the Hermitian inner product $\sum_j \boldsymbol{x}_j \bar{\boldsymbol{y}}_j$. Let T be an operator on $\mathbb{C}^n$ such that $\langle T\boldsymbol{x}, T\boldsymbol{y}\rangle = 0$ if $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0$. Prove that $T = kS$ for some scalar k and some operator S which is unitary (an isometry).

Solution: Let $A = T^*T$, where T^* is the adjoint of T. Then $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0$ implies $\langle A\boldsymbol{x}, \boldsymbol{y}\rangle = 0$. For any $\boldsymbol{x} \in \mathbb{C}^n$, let $x^\perp = \{y \in \mathbb{C}^n : \langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0\}$. So $\langle \boldsymbol{x}, \boldsymbol{x}'\rangle = 0$ for any $\boldsymbol{x}' \in x^\perp$. So $\langle A\boldsymbol{x}, \boldsymbol{x}'\rangle = 0$, which implies that $A\boldsymbol{x} \in (x^\perp)^\perp$, so $A\boldsymbol{x} = \lambda\boldsymbol{x}$ for some $\lambda \in \mathbb{C}$. Since every vector is an eigenvector of A, it follows that $A = rI$ for some scalar r. The constant r is a nonnegative real number since A is positive semidefinite. If $r = 0$ then $A = 0$, hence $T = 0$ and we may take $k = 0$, $S = I$. If $r > 0$, we take $k = \sqrt{r}$ and set $S = \frac{1}{\sqrt{r}}T$. Clearly k is real, and S is unitary because

$$S^*S = \frac{1}{\sqrt{k}}T * T\frac{1}{\sqrt{k}} = \frac{1}{k}A = I$$

Problem 4.

Consider the vector space $\mathcal{M}_{n\times n}(\mathbb{R})$ of $n \times n$ real matrices. Consider the linear map $T : \mathcal{M}_{n\times n}(\mathbb{R}) \longrightarrow \mathcal{M}_{n\times n}(\mathbb{R})$ given by $T(A) = A^T$ for all $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. Here A^T denotes the transpose of A.

- Find the characteristic polynomial and minimal polynomial of T.
- Find the Jordan form of T.

Solution:

- Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. Note that $T^2(A) = A$, so $T^2 - I = 0$. Thus, the minimal polynomial must divide $x^2 - 1 = (x-1)(x+1)$, so the only possible eigenvalues of T are 1 and -1. Moreover, both 1 and -1 are indeed eigenvalues, since $T(A) = A$ if A is symmetric, and $T(A) = -A$ if A is skew-symmetric A (i.e., if $A = -A^T$). Thus, the minimal polynomial is $m(x) = x^2 - 1$.

  Define $S = \left\{S^{(i,j)}, 1 \le i \le j \le n\right\}$ and $K = \left\{K^{(i,j)}, 1 \le i < j \le n\right\}$ by

  $$S^{(i,j)}_{k,\ell} = \begin{cases} 1 & \text{if } (k,\ell) = (i,j) \\ 1 & \text{if } (k,\ell) = (j,i) \\ 0 & \text{otherwise} \end{cases} \qquad K^{(i,j)}_{k,\ell} = \begin{cases} 1 & \text{if } (k,\ell) = (i,j) \\ -1 & \text{if } (k,\ell) = (j,i) \\ 0 & \text{otherwise.} \end{cases}$$

  Observe that $S \cup K$ forms a basis for $\mathcal{M}_{n\times n}(\mathbb{R})$. Moreover, each matrix in S is symmetric so is an eigenvector of T with eigenvalue 1, and each matrix in K is skew-symmetric so is an eigenvector with eigenvalue -1. Thus, T has a basis of eigenvectors, so is diagonalizable. Since T is diagonalize, the geometric multiplicity and the algebraic multiplicity of one of its given eigenvalue are equal and we will refer to them as the multiplicity of this given eigenvalue.

  Since S has $n(n+1)/2$ elements, the eigenvalue 1 has multiplicity $n(n+1)/2$. Similarly, the eigenvalue -1 has multiplicity $n(n-1)/2$. Thus, the characteristic polynomial is $p(x) = (x-1)^{n(n+1)/2}(x+1)^{n(n-1)/2}$.

- The Jordan form is

  $$J = \begin{bmatrix} I_S & 0 \\ 0 & -I_K \end{bmatrix}$$

  where I_S is an $n(n+1)/2 \times n(n+1)/2$ identity matrix and I_K is an $n(n-1)/2 \times n(n-1)/2$ identity matrix.

Part II. Work **two** of problems 5 through 8.

Problem 5.

Let A be an $n \times n$ matrix over $\mathbb{C}$. Show that

1. there exists an $n \times n$ invertible matrix Q such that

$$Q^{-1}AQ = \begin{bmatrix} \lambda_1 & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

2. Use mathematical induction to show that any $n \times n$ matrix over $\mathbb{C}$ is similar to an uppertriangular matrix

$$\begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Solution:

1. Let V be an n-dimensional vector space over $\mathbb{C}$. Let T be an operator on V. Suppose $\mathcal{M}(T, (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n)) = A$. Since V is complex, T must have an eigenvalue λ_1. Suppose $\boldsymbol{v}_1$ is the corresponding eigenvector. Add $n-1$ vectors $\boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ such that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ is a basis of V. Suppose

$$\begin{aligned} &T\boldsymbol{v}_1 = \lambda_1 \boldsymbol{v}_1 \\ &T\boldsymbol{v}_2 = b_{12}\boldsymbol{v}_1 + b_{22}\boldsymbol{v}_2 + \cdots + b_{n2}\boldsymbol{v}_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \\ &T\boldsymbol{v}_n = b_{1n}\boldsymbol{v}_1 + b_{2n}\boldsymbol{v}_2 + \cdots + b_{nn}\boldsymbol{v}_n \end{aligned} \tag{1}$$

Then

$$\mathcal{M}(T, (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n)) = \begin{bmatrix} \lambda_1 & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

Since A and B are different matrices of the same operator under different bases, they are similar. So there exists an invertible matrix Q such that

$$Q^{-1}AQ = \begin{bmatrix} \lambda_1 & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & b_{n2} & \cdots & b_{nn} \end{bmatrix} \tag{2}$$

2. The conclusion is true obviously for $n = 1$. Suppose the conclusion is true for $n-1$, now we show that it is true for n. Let A be any $n \times n$ matrix over n. From Part 1, we know there exists an invertible matrix Q_1 such that

$$Q_1^{-1} A Q_1 = \begin{bmatrix} \lambda_1 & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & b_{n2} & \cdots & b_{nn} \end{bmatrix}. \tag{3}$$

Let

$$B = \begin{bmatrix} b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots \\ b_{n2} & \cdots & b_{nn} \end{bmatrix} \tag{4}$$

Then there exists an $(n-1) \times (n-1)$ invertible matrix Q_2 such that

$$Q_2^{-1} B Q_2 = \begin{bmatrix} \lambda_2 & & & * \\ & \lambda_3 & & \\ & & \ddots & \\ & 0 & & \lambda_n \end{bmatrix} \tag{5}$$

Let

$$Q = Q_1 \begin{bmatrix} 1 & 0 \\ 0 & Q_2 \end{bmatrix}$$

so that

$$Q^{-1} A Q = \begin{pmatrix} 1 & 0 \\ 0 & Q_2^{-1} \end{pmatrix} \begin{bmatrix} \lambda_1 & b_{12} & \cdots & b_{1n} \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q_2 \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} \lambda_1 & * \\ 0 & Q_2^{-1} B Q_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \tag{7}$$

Problem 6.

Let V be a finite dimensional inner product space over $\mathbb{C}$. Let $T : V \to V$ be a self-adjoint operator on V. Suppose $\mu \in \mathbb{C}, \epsilon > 0$ are given and assume there is a unit vector $\boldsymbol{x} \in V$ such that

$$||T(x) - \mu \boldsymbol{x}|| \leq \epsilon$$

Show that there is an eigenvalue λ of T such that $|\lambda - \mu| \leq \epsilon$.

Solution: By the spectral theorem, there is an orthonormal basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ of V consisting of eigenvectors of T ($n = \dim(V)$). Let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues respectively. Then $\boldsymbol{x} = \sum_i \langle \boldsymbol{x}, \boldsymbol{e}_i \rangle \boldsymbol{e}_i$ and

$$T(\boldsymbol{x}) = \sum_{i=1}^{n} (\boldsymbol{x}, \boldsymbol{e}_i) \, T(\boldsymbol{e}_i) = \sum_{i=1}^{n} (\boldsymbol{x}, \boldsymbol{e}_i) \, \lambda_i \boldsymbol{e}_i \tag{8}$$

which leads to

$$\begin{aligned}
\|T(\boldsymbol{x}) - \mu \boldsymbol{x}\|^2 &= \left\| \sum_{i=1}^{n} (\boldsymbol{x}, \boldsymbol{e}_i) (\lambda_i - \mu) \boldsymbol{e}_i \right\|^2 \\
&= \left\langle \sum_{i=1}^{n} (\boldsymbol{x}, \boldsymbol{e}_i) (\lambda_i - \mu) \boldsymbol{e}_i, \sum_{j=1}^{n} (\boldsymbol{x}, \boldsymbol{e}_j) (\lambda_j - \mu) \boldsymbol{e}_j \right\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} (\boldsymbol{x}, \boldsymbol{e}_i) (\boldsymbol{x}, \boldsymbol{e}_j) (\lambda_i - \mu) (\lambda_j - \mu) (\boldsymbol{e}_i, \boldsymbol{e}_j) \\
&= \sum_{i=1}^{n} |(\boldsymbol{x}, \boldsymbol{e}_i)|^2 |\lambda_i - \mu|^2, \quad \text{by orthonormality}
\end{aligned} \tag{9}$$

If for every i, we had

$$|\lambda_i - \mu| > \epsilon$$

then

$$\|T(x) - \mu \boldsymbol{x}\|^2 > \varepsilon^2 \sum_{i=1}^{n} |(\boldsymbol{x}, \boldsymbol{e}_i)|^2 = \varepsilon^2 \|\boldsymbol{x}\| = \varepsilon^2 \tag{10}$$

However, this contradicts our assumption that $\|T(x) - \mu \boldsymbol{x}\|^2 \leq \varepsilon$. So there is some $i \in \{1, \ldots, n\}$ such that $|\lambda_i - \mu| \leq \epsilon$.

Problem 7.

Let V be an inner product space with inner product $\langle \cdot, \cdot \rangle$.

An operator P in $\mathcal{L}(V)$ is said to be a *projection* if and only if $P^2 = P$.

A projection P is said to be an *orthogonal projection* if and only if Range(P) and Null(P) are orthogonal subspaces.

1. Let P be a projection. Prove that P is an orthogonal projection if and only if P is diagonalizable in an orthogonal basis.

2. Let P be a projection. Prove that P is an orthogonal projection if and only if $\|Pv\| \leq \|v\|$ for every $v \in V$.

Solution:

(a) ($\Longrightarrow$) Let $P \in \mathcal{L}(V)$ be an orthogonal projection.

Any vector $v \in V$ can be written uniquely as $v = u + w$, where $u \in \text{Range}\, P$ and $w \in \text{null}\ P$. Since P is an orthogonal projection, $Pv = u$ and null $P = (\text{Range}\, P)^{\perp}$. Observe that if $v \in \text{Range}\, P$, then $Pv = v$, and if $v \in \text{null}\ P$, $Pv = 0v$. Thus, every vector in Range P is an eigenvector of P with corresponding eigenvalue 1, and every vector in null P is an eigenvector with corresponding eigenvalue 0. Let U be an orthogonal basis for Range P and let W be an orthogonal basis for null P. Since null $P = (\text{Range}\, P)^{\perp}$, every vector in U is orthogonal to every vector in W. Thus, $U \cup W$ is an orthogonal basis of eigenvectors for V. Hence, P is diagonalizable in an orthogonal basis.

($\Longleftarrow$) Let P is a projection that is diagonalizable in an orthogonal basis. Since P is projection, then $P^2 = P$. So $P(P - I) = 0$. So the eigenvalues of P are either (only 0) or (0 and 1) or (only 1). If (only 0), then $P = 0$, so Range$(P) = \{0\}$ and Null$(P) = V$, so Range(P) and Null(P) are orthogonal subspaces. If (only 1), then $P = I$, so Range$(P) = V$ and Null$(P) = \{0\}$, so Range(P) and Null(P) are orthogonal subspaces. If (0 and 1), we consider E_0 and E_1 the associated eigenspaces of P. First, since P is diagonalizable in an orthogonal basis, E_0 and E_1 are orthogonal subspaces. Second since $E_0 = \text{Null}(P)$ and $E_1 = \text{Range}(P)$, then Range(P) and Null(P) are orthogonal subspaces.

(b) ($\Longrightarrow$) Suppose P is an orthogonal projection. Then for any $v \in V$, $v = Pv + w$, with $\langle Pv, w \rangle = 0$. Thus,

$$\|v\|^2 = \langle Pv + w, Pv + w \rangle = \|Pv\|^2 + \|w\|^2 \geq \|Pv\|^2 .$$

Hence, $\|Pv\| \leq \|v\|$.

($\Longleftarrow$) We prove the contrapositive. Suppose P is not an orthogonal projection. Then null $P \neq (\text{Range}\, P)^{\perp}$. Thus, there exist $u \in \text{Range}\, P$ an $w \in$ null P such that $\langle u, w\rangle > 0$. Let $v = u - \alpha w$ for some $\alpha > 0$. Then

$$\|v\|^2 = \langle u - \alpha w, u - \alpha w\rangle = \|u\|^2 - 2\alpha \langle u, w\rangle + \alpha^2 \|w\|^2 .$$

Choosing $\alpha \in \left(0, \frac{\|w\|^2}{2\langle u,w\rangle}\right)$ yields

$$\|v\|^2 < \|u\|^2 = \|Pv\|^2 .$$

Thus, if $\|Pv\| \leq \|v\|$ for all $v \in V$, then P is an orthogonal projection.

Problem 8.

Recall that trace$(A^T B)$ defines an inner product on $\mathcal{M}_{n\times n}(\mathbb{R})$ and that the norm associated with this inner product is $\|\cdot\|_{\text{fro}}$, the Frobenius norm of a matrix. Recall that a symmetric matrix S is such that $S = S^T$ and a skew-symmetric matrix N is such that $N = -N^T$.

Let $\mathcal{S}$ be the subspace of symmetric matrices in $\mathcal{M}_{n\times n}(\mathbb{R})$. Let $\mathcal{N}$ the subspace of skew-symmetric matrices in $\mathcal{M}_{n\times n}(\mathbb{R})$.

1. Prove that
$$\mathcal{S} \oplus \mathcal{N} = \mathcal{M}_{n\times n}(\mathbb{R})$$
2. Prove that the subspaces $\mathcal{S}$ and $\mathcal{N}$ are orthogonal with respect to the inner product trace$(A^T B)$.
3. Let A be in $\mathcal{M}_{n\times n}(\mathbb{R})$. Prove that
$$\min_{S\in\mathcal{S}} \|A - S\|_{\text{fro}} = \frac{1}{2}\|A - A^T\|_{\text{fro}}.$$

Solution:

1. Let A in $\mathcal{M}_{n\times n}(\mathbb{R})$. Then
$$A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right)$$
. We note that $\frac{1}{2}\left(A + A^T\right)$ is symmetric, so is in $\mathcal{S}$. We note that $\frac{1}{2}\left(A - A^T\right)$ is skew-symmetric, so is in $\mathcal{N}$. This shows that
$$\mathcal{S} + \mathcal{N} = \mathcal{M}_{n\times n}(\mathbb{R})$$

Let A in $\mathcal{S} \cap \mathcal{N}$. Then

$$A = \frac{1}{2}\left(A + A^T\right) \quad A = \frac{1}{2}\left(A - A^T\right).$$

We see that this implies

$$\left(A + A^T\right) = \left(A - A^T\right).$$

This implies

$$2A^T = 0.$$

This implies

$$A = 0.$$

So that

$$\mathcal{S} \cap \mathcal{N} = \{0\}.$$

So $\mathcal{S} + \mathcal{N} = \mathcal{M}_{n\times n}(\mathbb{R})$ and $\mathcal{S} \cap \mathcal{N} = \{0\}$, so

$$\mathcal{S} \oplus \mathcal{N} = \mathcal{M}_{n\times n}(\mathbb{R}).$$

2. Let S be symmetric and N be skew-symmetric then

$$\begin{array}{rcll}
\operatorname{trace}(S^TN) & = & \operatorname{trace}((S^TN)^T) & \text{because } \operatorname{trace}(A) = \operatorname{trace}(A^T); \\
& = & \operatorname{trace}(N^T(S^T)^T) & \text{because } (AB)^T = (B^TA^T); \\
& = & \operatorname{trace}(N^TS) & \text{because } (A^T)^T = (A); \\
& = & \operatorname{trace}(-NS) & \text{because } N \text{ is skew-symmetric, } N = -N^T; \\
& = & -\operatorname{trace}(NS) & \text{because } \operatorname{trace}(-A) = -\operatorname{trace}(A); \\
& = & -\operatorname{trace}(SN) & \text{because } \operatorname{trace}(AB) = \operatorname{trace}(BA); \\
& = & -\operatorname{trace}(S^TN) & \text{because } S \text{ is symmetric, } S = S^T;
\end{array}$$

So we find $\operatorname{trace}(S^TN) = -\operatorname{trace}(S^TN)$, which means $\operatorname{trace}(S^TN) = 0$.

Therefore $\mathcal{S} \perp \mathcal{N}$.

We can also take cannonical basis for $\mathcal{S}$ and cannonical basis for $\mathcal{N}$ and prove orthogonality.

3. We define the orthogonal projection operator $P_{\mathcal{S}}$ on the set of symmetric matrixes. We define the orthogonal projection operator $P_{\mathcal{N}}$ on the set of skew-symmetric matrixes.

   Since $\mathcal{S}$ and $\mathcal{N}$ are orthogonal supplement subspaces, (from question 8.2,) we have

$$P_{\mathcal{S}} + P_{\mathcal{N}} = I.$$

   From question (8.1) and (8.2), we understand that, for any matrix A in $\mathcal{M}_{n\times n}(\mathbb{R})$,

$$P_{\mathcal{S}}(A) = \frac{1}{2}\left(A + A^T\right).$$

$$P_{\mathcal{N}}(A) = \frac{1}{2}\left(A - A^T\right).$$

When a norm, $\|\cdot\|$, is induced by an inner product, we also know that

$$\min_{S \in \text{“subspace”}} \|A - S\| = \|A - P_{\text{orthogonal projection on “subspace”}}(A)\|.$$

Replacing “subpsace” by $\mathcal{S}$, and the norm by $\|\cdot\|_{\text{fro}}$, we get

$$\min_{S \in \mathcal{S}} \|A - S\|_{\text{fro}} = \|A - P_{\mathcal{S}}(A)\|_{\text{fro}}.$$

And so

$$\min_{S \in \mathcal{M}_{n\times n}(\mathbb{R}) \text{ such that } S = S^T} \|A - S\|_{\text{fro}} = \frac{1}{2}\|A - A^T\|_{\text{fro}}.$$
