University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
July 12, 2019

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to complete all six problems.
- Each problem is worth 20 points
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Notation: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{n,n}$ are the vector spaces of n-tuples and $n \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- Ask the proctor if you have any questions.

Good luck!

1.	______	4.	______
2.	______	5.	______
3.	______	6.	______
		Total	______

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
Steve Billups, Steffen Borgwardt (Chair), Yaning Liu.

Problem 1.

a. (6 points) Prove or reject:

   There exists a matrix $A \in \mathbb{R}^{4\times 4}$ for which the column space and null space are identical.

b. (9 points) Let $A \neq 0$ be an $m \times n$ matrix with $m \leq n$, let $b \in \mathbb{R}^m$ such that $Ax = b$ has no solution, and let $d \neq 0 \in \mathbb{R}^m$ for which there exists a solution to $Ax = d$.

   What is the minimal and maximal dimension of the set of solutions for $Ax = d$? Provide the best bounds available based on the given information, prove that your bounds are correct, and prove that they can be tight for all well-defined m, n.

c. (5 points) Suppose that S is a fixed, invertible $n \times n$ matrix. Let W be the set of all matrices A for which $S^{-1}AS$ is diagonal.

   Prove or reject: W is a vector space.

Problem 2.

a. (4 points) Let $T : \mathbb{P}_3 \to \mathbb{P}_3$ be an operator that maps $p(t) = a_0 + a_1 t^1 + a_2 t^2 + a_3 t^3$ onto $q(t) = a_3 t^1 + a_2 t^2 + a_1 t^3$.

   Prove or reject: T is a linear transformation. If so, provide a matrix representation.

b. (7 points) The first four Hermite polynomials are

$$1, \quad 1 - t, \quad -2 + 4t^2, \quad -12t + 18t^3.$$

   They form a basis β of $\mathbb{P}_3$, the space of polynomials of degree at most 3.

   Compute the change-of-coordinates matrix $P_{\beta \to \gamma}$ from β to a new basis γ of $\mathbb{P}_3$ given by

$$t^3 + t^2 + 2t, \quad t^2 + 2t, \quad 1 + t, \quad t.$$

   (Hint: $P_{\beta \to \gamma}$, when multiplied with a coordinate vector with respect to β gives a coordinate vector with respect to γ.)

c. (9 points) Let $a, b \neq 0 \in \mathbb{R}$ be fixed. Find a basis for the subspace in $\mathbb{R}^4$ created from intersecting

$$S = \operatorname{span}\left\{ \begin{pmatrix} a \\ 0 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} a \\ 0 \\ a \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ b \\ 0 \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad T = \operatorname{span}\left\{ \begin{pmatrix} b \\ 0 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} 0 \\ a \\ b \\ 0 \end{pmatrix} \right\}.$$

Problem 3.

Let V be a finite-dimensional vector space.

a. (7 points) Suppose $T \in \mathcal{L}(V)$ is such that every vector in V is an eigenvector of T. Prove or disprove that T is a scalar multiple of the identity operator.

b. (13 points) Suppose $T \in \mathcal{L}(V)$ is such that every subspace of V with dimension $\dim V - 1$ is invariant under T. Prove that T is a scalar multiple of the identity operator.

Problem 4.

Let $\|\cdot\|$ denote an arbitrary vector norm on $\mathbb{R}^p$. The matrix norm induced by $\|\cdot\|$ is defined by

$$\|P\| = \max_{x \neq 0} \frac{\|Px\|}{\|x\|}$$

for each $p \times p$ real matrix P.

a. (7 points) Prove that $\|\cdot\|$ is a norm on the vector space of real $p \times p$ matrices.

b. (13 points) Let P be a $p \times p$ real matrix. Suppose that $\|P\| < 1$. Prove that $I + P$ is nonsingular and that

$$\frac{1}{1 + \|P\|} \leq \|(I + P)^{-1}\| \leq \frac{1}{1 - \|P\|}.$$

Problem 5.

Let V be an n-dimensional inner product space over $\mathbb{F}$.

a. (5 points) Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove or reject: $U^{\perp}$ is invariant under T^{*} if U is invariant under T.

b. (5 points) Let T_1 and T_2 be two self-adjoint operators on V. Prove or reject: $T_1T_2 + T_2T_1$ is also self-adjoint.

c. (10 points) Let T be a self-adjoint operator on V. Show that T is a nonnegative self-adjoint operator on V if and only if the eigenvalues of T are all nonnegative real numbers.

Problem 6.

a. (6 points) Let $A \in \mathbb{F}^{n,n}$ be a square matrix that satisfies $A^2 = A$. Show that A is similar to the diagonal matrix.

$$C = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} = \begin{bmatrix} I_r & \\ & 0 \end{bmatrix}$$

That is, I_r is an identity square block of order r, $0 \leq r \leq n$.

b. (6 points) Let $A \in F^{n,n}$, $B \in F^{n,n}$ be square matrices such that $A^2 = A$, $B^2 = B$, and $AB = BA$. Suppose P_0 is an invertible matrix such that

$$P_0^{-1} A P_0 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Let $B_0 = P_0^{-1} B P_0$. Show that B_0 is in the form of

$$B_0 = \begin{bmatrix} B_1 & \\ & B_2 \end{bmatrix}$$

where B_1 is of order r, and $B_1^2 = B_1$ and $B_2^2 = B_2$.

c. (8 points) Let $A \in F^{n,n}$, $B \in F^{n,n}$ be square matrices such that $A^2 = A$, $B^2 = B$, and $AB = BA$. Show that there exists an invertible matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are both diagonal, and the diagonal entries are 0 and 1 for both. *(Hint: Let P_0 be the invertible matrix for A in part (b). Let Q_1 and Q_2 be invertible matrices that serve the same role for B_1 and B_2, respectively. Use P_0, Q_1 and Q_2 to construct the matrix P.)*