University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
June 17, 2016

Name: ____

Exam Rules:

- This exam lasts 4 hours.
- There are 8 problems. Each problem is worth 20 points. All solutions will be graded and your final grade will be based on your six best problems. Your final score will count out of 120 points.
- You are not allowed to use books or any other auxiliary material on this exam.
- Start each problem on a separate sheet of paper, write only on one side, and label all of your pages in consecutive order (*e.g.*, use 1-1, 1-2, 1-3, ..., 2-1, 2-2, 2-3, ...).
- Read all problems carefully, and write your solutions legibly using a dark pencil or pen in "essay-style" using full sentences and correct mathematical notation.
- Justify your solutions: cite theorems you use, provide counterexamples for disproof, give clear but concise explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, you may not merely quote or rephrase that theorem as your solution; instead, you must produce an independent proof.
- If you feel that any problem or any part of a problem is ambiguous or may have been stated incorrectly, please indicate your interpretation of that problem as part of your solution. Your interpretation should be such that the problem is not trivial.
- Please ask the proctor if you have any other questions.

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
Varis Carey, Stephen Hartke, and Julien Langou (Chair).

Problem 1

Let P be the vector space of single-variable polynomials over the reals of degree at most 4. Let D be the differential operator.

(a) Prove that D is a linear operator on P.

(b) Determine the rank and nullity of D as a linear operator on P. Find bases for the nullspace of D and the image of D.

Problem 2

Let $V = \mathbb{R}^m$, and let W be a subspace of V with basis $\{x_1, x_2, \dots, x_n\}$. Let v be any vector in V. Derive the normal equations method for computing the best approximation $w \in W$ to v.

Problem 3

Let V be a real vector space, let A and B be two subspaces of V, let $\tilde{A}$ be a subspace of V such that $\tilde{A} \oplus (A \cap B) = A$ and let $\tilde{B}$ be a subspace of V such that $\tilde{B} \oplus (A \cap B) = B$. Prove that $A + B = (A \cap B) \oplus \tilde{A} \oplus \tilde{B}$.

Problem 4

Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Define

$$\begin{aligned} T : \mathcal{M}_2(\mathbb{R}) &\longrightarrow \mathcal{M}_2(\mathbb{R}) \\ B &\longmapsto AB - BA. \end{aligned}$$

(a) Fix an ordered basis $\mathcal{B}$ of $\mathcal{M}_2(\mathbb{R})$ and compute the matrix $[T]_{\mathcal{B}}$ that represents T with respect to this basis.

(b) Compute a basis for each of the eigenspaces of T.

Problem 5

For arbitrary complex scalars a, b, and c, compute the minimal polynomial of the matrix

$$A = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix}.$$

Problem 6

Let $V = \mathbb{P}(\mathbb{R})$, and let U be a subspace of V given by $\text{span}(1, x, x^2)$.

(a) Pick a basis for U, and find the corresponding dual basis.

(b) Given the inner product on $\langle p_1, p_2\rangle = \int_0^1 p_1(x)p_2(x)\,dx$, find the Riesz representers (in U) of the dual basis in part a). Recall that the Riesz representer is the unique vector u in V s.t., given a fixed ϕ in V', $\phi(v) = \langle v, u\rangle$.

(a) We pick the standard basis for $U(u_1, u_2, u_3 = 1, x, x^2)$. The dual basis ϕ_i in U', the dual space of U, are functionals on U s.t. $\phi_i(u_j) = \delta_{ij}$. What is an example of a functional on U s.t. $\phi(1) = 1$ but $\phi(x) = \phi(x^2) = 0$? $\phi_1(p) = p(0)$. The derivative operator, which we encountered in the first problem, works for $\phi_2 = Dp(0)$. Finally, for $\phi_3 = 1/2D^2p(0)$. (In general, the dual basis of a monomial basis will be the corresponding Taylor series coefficient).

(b) The easy way to find the Riesz representors for ϕ_i is to find an orthonormal basis for U and then use the result:

$$u_{\phi_i} = \sum_j \phi_i(o_j)o_j$$

We apply Gram-Schmidt to $\{1, x, x^2\}$.

$o_1 = 1;$

$$o_2 = \frac{x - \langle 1, x\rangle 1}{\|x - \langle 1, x\rangle 1\|} = \frac{(x - 1/2)}{\|x - 1/2\|} = \sqrt{6}(x - 1/2)$$

$$p_3 = x^2 - \langle 1, x^2\rangle - \langle \sqrt{6}(x - 1/2), x^2\rangle\sqrt{6}(x - 1/2); \quad o_3 = \frac{p_3}{\|p_3\|}$$

Need to compute o_3, plut in to compute u_{ϕ_i}

There is another way of solving most of the probem. If the choose an orthonormal basis for U(by performing G-S on $\{1, x, x^2\}$ then part b) s trivial-the Riesz representor of the the dual basis is just the orthonormal basis as $\langle o_j, o_i\rangle = \delta_{ij}$. However, it is very difficult to construct the corresponding functional of that representor. If the students followed this approach, stating the definitions of the dual basis but getting stuck, I'd probably award 15 out of 20 points.

Problem 7

Suppose that A and B are two symmetric real $n \times n$ matrices and that A is positive definite. Show that there is an invertible real matrix U such that $U^T AU$ is the identity matrix and $U^T BU$ is diagonal.

Problem 8

Let V be a finite-dimensional real vector space.

(a) Suppose $T \in \mathcal{L}(V)$ and m is a nonnegative integer such that

$$\text{Range } T^m = \text{Range } T^{m+1}.$$

Prove that Range $T^k =$ Range T^m for all $k > m$.

(b) Prove or give a counterexample: if $T \in \mathcal{L}(V)$, then

$$V = \text{Null } T \oplus \text{Range } T.$$

(c) Prove that if $T \in \mathcal{L}(V)$, then

$$V = \text{Null } T^n \oplus \text{Range } T^n,$$

where $n = \dim V$.