University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
May 25th, 2018

Name: ____

Exam Rules:

- This exam lasts 4 hours.
- There are 8 problems. Each problem is worth 20 points. All solutions will be graded and your final grade will be based on your *six best problems*. Your final score will be out of 120 points.
- You are not allowed to use books or any other auxiliary material on this exam.
- Start each problem on a separate sheet of paper, write only on one side, and label all of your pages in consecutive order (*e.g.*, use 1-1, 1-2, 1-3, . . . , 2-1, 2-2, 2-3, . . .).
- Read all problems carefully, and write your solutions legibly using a dark pencil or pen in "essay-style" using full sentences and correct mathematical notation.
- Justify your solutions: cite theorems you use, provide counterexamples for disproof, give clear but concise explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, you may not merely quote or rephrase that theorem as your solution; instead, you must produce a complete proof.
- Parts of a multipart question are not necessarily worth the same number of points.
- If you feel that any problem or any part of a problem is ambiguous or may have been stated incorrectly, please indicate your interpretation of that problem as part of your solution. Your interpretation should be such that the problem is not trivial.
- Please ask the proctor if you have any questions.

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2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
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Applied Linear Algebra Preliminary Exam Committee:
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Problem 1

Let $V = \mathbb{P}^2(\mathbb{R})$, the space of real-valued polynomials of total degree less than or equal to 2. Let $S = \{x, 1-x, 1-x^2\}$ be a set of vectors in V.

(a) Show that S is a basis of V.

(b) Let $T : V \to V$ given by $p(x) \to xp'(x)$. Find the matrix of T with respect to S.

(c) Is T invertible? If not, express its nullspace in terms of S.

Solution:

(a) In terms of the standard basis $(1, x, x^2)$ and writing the vectors in S as the columns of a matrix yield:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Exchanging row 1 and row 2 yields a matrix in upper triangular form with a pivot in every column, so the columns of S are linearly independant, and as the dimension of $\mathbb{P}^2$ is 3, we have a basis.

(b) $T(x) = x \to [1,0,0]^\top$; $T(1-x) = T(1) - T(x) = [0,0,0] + [-1,0,0]$; $T(1-x^2) = T(1) - T(x^2) = -2x^2 = -2S_1 - 2S_2 + 2S_3$. So

$$\mathcal{M}_T = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 2 \end{pmatrix}.$$

(c) $\mathcal{M}_T$ has linearly dependant columns, so T is not invertible. Any constant function will get mapped to the 0 polynomial, so the nullspace of T (one-dimensional as $\mathcal{M}_T$ has two pivots) is equal to $c(S_1 + S_2)$, $c \in \mathbb{R}$.

Problem 2

Suppose $V_1, V_2, \ldots, V_s$ are subspaces of V.

(a) Show that the sum of $V_1, V_2, \ldots, V_s$ is a direct sum if and only if

$$V_i \cap \sum_{j \neq i} V_j = \mathbf{0}, i = 1, \ldots, s.$$

(b) Show that the sum of $V_1, V_2, \ldots, V_s$ is a direct sum if and only if

$$V_1 \cap V_2 = \mathbf{0}, (V_1 + V_2) \cap V_3 = \mathbf{0}, \ldots, (V_1 + V_2 + \cdots + V_{s-1}) \cap V_s = \mathbf{0}.$$

Solution:

(a) Suppose

$$V_i \cap \sum_{j \neq i} V_j = \mathbf{0}, i = 1, \ldots, s.$$

Let $\boldsymbol{v}_k \in V_k, k = 1, \ldots, s$ such that

$$\boldsymbol{v}_1 + \boldsymbol{v}_2 + \cdots + \boldsymbol{v}_i + \cdots + \boldsymbol{v}_s = \mathbf{0}.$$

Then for each $i = 1, \ldots, s$

$$\boldsymbol{v}_i = -\boldsymbol{v}_1 - \cdots - \boldsymbol{v}_{i-1} - \boldsymbol{v}_{i+1} - \cdots - \boldsymbol{v}_s \in V_i \cap \sum_{j \neq i} V_j = \mathbf{0}.$$

So $\boldsymbol{v}_i = \mathbf{0}, i = 1, \ldots,$. So $\sum_{i=1}^{s} V_i$ is a direct sum.

Now suppose $\sum_{i=1}^{s} V_i$ is a direct sum. For each i and any $\boldsymbol{v} \in V_i \cap \sum_{j \neq i} V_j$, we have

$$\boldsymbol{v} = \boldsymbol{v}_1 + \cdots + \boldsymbol{v}_{i-1} + \boldsymbol{v}_{i+1} + \cdots + \boldsymbol{v}_s \in V_i.$$

So

$$\boldsymbol{v}_1 + \cdots + \boldsymbol{v}_{i-1} + (-\boldsymbol{v}) + \boldsymbol{v}_{i+1} + \cdots + \boldsymbol{v}_s = \mathbf{0}.$$

Since $\sum_{i=1}^{s} V_i$ is a direct sum, $\boldsymbol{v} = 0$. So $V_i \cap \sum_{j \neq i} V_j = \mathbf{0}, i = 1, \ldots, s$.

(b) Suppose $V_1 + \cdots + V_s$ is a direct sum, then based on Part (a), we have

$$V_i \cap \sum_{j \neq i} V_j = \mathbf{0}, i = 1, \ldots, s.$$

For each i, it is obvious that

$$V_i \cap \sum_{j < i} V_j \subseteq V_i \cap \sum_{j \neq i} V_j.$$

So

$$V_i \cap \sum_{j < i} V_j = \mathbf{0}, i = 1, \ldots, s.$$

On the other hand, we use induction to show the sufficiency. For $s = 2$, $V_1 \cap V_2 = \mathbf{0}$ obviously shows $V_1 + V_2$ is a direct sum. Now suppose the sufficiency holds for $s = k$, and we will show it also holds for $s = k + 1$. Suppose now we have

$$V_1 \cap V_2 = \mathbf{0}, (V_1 + V_2) \cap V_3 = \mathbf{0}, \ldots, (V_1 + V_2 + \cdots + V_k) \cap V_{k+1} = \mathbf{0}.$$

Based on the induction, the first $k-1$ conditions above suggest $V_1 + \cdots + V_k$ is a direct sum. In addition, the last condition above shows that $(V_1 + \cdots + V_k) + V_{k+1}$ is a direct sum. Combining the two facts, it is easy to show that $V_1 + \cdots + V_k + V_{k+1}$ is a direct sum.

Problem 3

Let A be a real matrix satisfying $A^3 = A$.

(a) Prove that A can be diagonalized.

(b) If A is a 3×3 matrix, how many different possible similarity classes are there for A?

Solution:

(a) Note that A satisfies $A^3 - A = 0$. Hence, the minimal polynomial of A divides $x^3 - x = x(x-1)(x+1)$, which factors into linear factors. Each Jordan block in the Jordan canonical form of A is 1×1, since each eigenvalue is the root of a linear factor of the minimal polynomial. Thus, the Jordan canonical form of A is a diagonal matrix.

(b) Since the minimal polynomial divides $x^3 - x$, the possible eigenvalues of A are 0, 1, and -1. Let t_λ denote the number of Jordan blocks for eigenvalue λ (equivalently, t_λ is the geometric multiplicity of λ). Then $t_0 + t_1 + t_{-1} = 3$, where each t_λ is a nonnegative integer. There are 10 solutions for (t_0, t_1, t_{-1}):

$$\begin{array}{lll} (3,0,0) & (0,1,2) & (1,1,1) \\ (0,3,0) & (0,2,1) & \\ (0,0,3) & (1,0,2) & \\ & (1,2,0) & \\ & (2,0,1) & \\ & (2,1,0) & \end{array}$$

Alternatively, a combinatorial stars-and-bars argument shows there are $\binom{5}{2} = 10$ solutions. Thus, there are 10 different possible similarity classes.

Problem 4

Let T be a self-adjoint operator on an n-dimensional inner product space V. Let λ_0 be an eigenvalue of T. Show that the (algebraic) multiplicity of λ_0 equals dim $E(\lambda_0, T)$, i.e., the dimension of the eigenspace of T corresponding to λ_0.

Solution: Let m be the algebraic multiplicity of λ_0. Then $m \geq \dim\, E(\lambda_0, T)$, since m is defined to be the dimension of the generalized eigenspace of T corresponding to λ_0, which is greater than or equal to dim $E(\lambda_0, T)$.

Next we show that $m \leq \dim\, E(\lambda_0, T)$. Since T is self-adjoint, based on the Spectral Theorems, there exists an orthonormal basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m$ such that

$$\mathcal{M}(T, (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m)) = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

So the diagonal elements are all the eigenvalues of T. WLOG, let $\lambda_1 = \cdots = \lambda_m = \lambda_0$. Then

$$T\boldsymbol{e}_i = \lambda_0 \boldsymbol{e}_i, i = 1, \ldots, m.$$

So $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m$ are m linearly independent vectors in $E(\lambda_0, T)$ and dim $E(\lambda_0, T) \geq m$.
As a result, we have $m = \dim\, E(\lambda_0, T)$.

Problem 5

Let T and S be linear maps on inner product space V. For any vector $\boldsymbol{v} \in V$,

$$\langle T\boldsymbol{v}, T\boldsymbol{v}\rangle = \langle S\boldsymbol{v}, S\boldsymbol{v}\rangle .$$

Show that range T and range S are isomorphic.

Solution: Let

$$\varphi : T\boldsymbol{v} \to S\boldsymbol{v}$$

for any $\boldsymbol{v} \in V$. We show that φ is an isomorphism between range T and range S.
First, the map φ is well-defined. Suppose $T\boldsymbol{v} = T\boldsymbol{w}$, $\boldsymbol{w} \in V$. Then $T(\boldsymbol{v} - \boldsymbol{w}) = \mathbf{0}$. So

$$\langle S(\boldsymbol{v} - \boldsymbol{w}), S(\boldsymbol{v} - \boldsymbol{w})\rangle = \langle T(\boldsymbol{v} - \boldsymbol{w}), T(\boldsymbol{v} - \boldsymbol{w})\rangle = \langle \mathbf{0}, \mathbf{0}\rangle = 0.$$

Hence $S(\boldsymbol{v} - \boldsymbol{w}) = \mathbf{0}$, and $S\boldsymbol{v} = S\boldsymbol{w}$.
The map is also a linear map, since

$$\begin{aligned}\varphi(T\boldsymbol{v} + T\boldsymbol{w}) &= \varphi(T(\boldsymbol{v} + \boldsymbol{w})) = S(\boldsymbol{v} + \boldsymbol{w}) = S\boldsymbol{v} + S\boldsymbol{w} \\ \varphi(\lambda T\boldsymbol{v}) &= \varphi(T(\lambda\boldsymbol{v})) = S(\lambda\boldsymbol{v}) = \lambda S\boldsymbol{v}.\end{aligned}$$

Finally, it is obvious that φ is surjective. In addition, suppose $T\boldsymbol{v} \neq T\boldsymbol{w}$. Then $S\boldsymbol{v} \neq S\boldsymbol{w}$. Otherwise if $S\boldsymbol{v} = S\boldsymbol{w}$, $T\boldsymbol{v} = T\boldsymbol{w}$. So φ is injective. So φ is an isomorphism between range T and range S.

Problem 6

Let V be a complex-valued vector space.

(a) Give an example of an operator T that is surjective but not invertible.

(b) Given an example of an operator S that has no eigenvalue.

(c) Let $V = \mathbb{C}^\infty$. Let U be the set of vectors in V with finitely many non-zero entries. Prove that U is an infinite-dimensional subspace of V.

Solution:

(a) As T is a surjective operator, but not invertible, V must be infinite dimensional. As T is not invertible, it must have a non-trivial nullspace. Consider the vector space of infinite lists, $\mathbb{C}^\infty$. The left-shift operator $T : V \to V$, $(z_1, z_2, \dots,) \to (z_2, z_3, \dots)$ is clearly surjective as $T(z_1, z_1, z_2, \dots) = (z_1, z_2, \dots)$, but it is not injective as $(z_1, 0, 0, \dots) \to 0_V$. T is linear, is trivial to check.

(b) Again, as we are over the complex field, S must be infinite dimensional. Consider the right-shift operator $S : V \to V$, $(z_1, z_2, \dots) \to (0, z_1, z_2, \dots)$. Let $Sv = \lambda v$. Then $0 = \lambda v_1$. Thus $\lambda = 0$ or $v_1 = 0$. If $\lambda = 0$ then $v = 0_V$, so v is not an eigenvector by definition. If $v_1 = 0$, then $(Sv)_2 = 0$, which (as $\lambda \neq 0$) now implies that $v_2 = 0$. By induction, $v = 0_v$, so again, v is not an eigenvector, and therefore S does not have an eigenvalue.

(c) U is a subspace of V as ku will have finitely many non-zero entries, the vector of all zeros has finitely any non-zero entries, and if u_1 has M non-zeros and u_2 has N nonzero entries then $u_1 + u_2$ has at most $M + N$ nonzeros, also finite, and hence $u_1 + u_2 \in U$.

Clearly e_i is in U, and e_i cannot be written as a linear combination of e_j, $j \neq i$. We have an infinite linear independant set in U, so U is infinite-dimensional.

Problem 7

Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times m}$. Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Solution: Let e_i be a unit vector in $\mathbb{R}^q$. $C_{ij} =< e_i, Ce_j >$, so for square C, $\mathrm{tr}(C) = \sum_{i=1}^{n} < e_i, Ce_i >$.
AB is $m \times m$, so we have $\mathrm{tr}(AB) = \sum_{i=1}^{m} < e_i, ABe_i >$. Note $< e_i, ABe_i >=< A^\top e_i, Be_i >$. Be_i is a column vector whose entries are the i-th column of B, i.e. $(B_{1i}, B_{2i}, \ldots, B_{ni})^\top$. $A^\top e_i$ is a column vector whose entries are the i-th column of $A^\top = (A^\top_{1i}, A^\top_{2i}, \ldots, A^\top_{ni}) = (A_{i1}, A_{i2}, \ldots, A_{in})$. The inner product of these two vectors is given by $\sum_{j=1}^{n} A_{ij} B_{ji}$. Combining, we have

$$\sum_{i=1}^{m} < e_i, ABe_i >= \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} B_{ji}.$$

We also have (by an almost identical argument)

$$\mathrm{tr}(BA) = \sum_{j=1}^{n} < e_j, BAe_j >= \sum_{j=1}^{n} \sum_{i=1}^{m} B_{ji} A_{ij}.$$

Switching the order of summation and using the commutativity of the reals finishes the proof.

Problem 8

Let W_1 and W_2 be two subspaces of an n-dimensional vector space V, and $\dim W_1 + \dim W_2 = n$. Show that there exists an operator T on V such that

$$\text{null } T = W_1 \quad \text{and} \quad \text{range } T = W_2.$$

Solution: Suppose $\dim W_1 = s$ and $\dim W_2 = m = n - s$. Let a basis of W_2 be

$$\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m.$$

Let a basis of W_1 be

$$\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_s,$$

and extend it to a basis of V:

$$\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_s, \boldsymbol{w}_{s+1}, \ldots, \boldsymbol{w}_n.$$

Let T be the operator defined as

$$T\boldsymbol{w}_1 = \mathbf{0}, T\boldsymbol{w}_2 = \mathbf{0}, \ldots, T\boldsymbol{w}_s = \mathbf{0}, T\boldsymbol{w}_{s+1} = \boldsymbol{v}_1, T\boldsymbol{w}_{s+2} = \boldsymbol{v}_2, \ldots, T\boldsymbol{w}_n = \boldsymbol{v}_m.$$

It is obvious that the operator exists.
Note that

$$\text{range } T = \text{span } (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m) = W_2.$$

In addition, it is obvious that

$$W_1 = \text{span } (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_s) \subset \text{null } T.$$

Let any $\boldsymbol{v} \in V$ and

$$\boldsymbol{v} = k_1\boldsymbol{w}_1 + \cdots + k_s\boldsymbol{w}_s + k_{s+1}\boldsymbol{w}_{s+1} + \cdots + k_n\boldsymbol{w}_n.$$

Then

$$\begin{aligned} T\boldsymbol{v} &= T(k_1\boldsymbol{w}_1 + \cdots + k_s\boldsymbol{w}_s + k_{s+1}\boldsymbol{w}_{s+1} + \cdots + k_n\boldsymbol{w}_n) \\ &= k_{s+1}T\boldsymbol{w}_{s+1} + \cdots + k_nT\boldsymbol{w}_n = \mathbf{0} \end{aligned}$$

suggests $k_{s+1} = \cdots = k_n = 0$. So $\boldsymbol{v} \in W_1$, which means null $T \subset W_1$. So null $T = W_1$.