University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
June 17, 2016

Name: ____

Exam Rules:

- This exam lasts 4 hours.
- There are 8 problems. Each problem is worth 20 points. All solutions will be graded and your final grade will be based on your six best problems. Your final score will count out of 120 points.
- You are not allowed to use books or any other auxiliary material on this exam.
- Start each problem on a separate sheet of paper, write only on one side, and label all of your pages in consecutive order (*e.g.*, use 1-1, 1-2, 1-3, . . . , 2-1, 2-2, 2-3, . . .).
- Read all problems carefully, and write your solutions legibly using a dark pencil or pen in "essay-style" using full sentences and correct mathematical notation.
- Justify your solutions: cite theorems you use, provide counterexamples for disproof, give clear but concise explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, you may not merely quote or rephrase that theorem as your solution; instead, you must produce an independent proof.
- If you feel that any problem or any part of a problem is ambiguous or may have been stated incorrectly, please indicate your interpretation of that problem as part of your solution. Your interpretation should be such that the problem is not trivial.
- Please ask the proctor if you have any other questions.

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2.	____________	6.	____________
3.	____________	7.	____________
4.	____________	8.	____________
		Total	____________

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Applied Linear Algebra Preliminary Exam Committee:
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Problem 1

Let P be the vector space of single-variable polynomials over the reals of degree at most 4. Let D be the differential operator.

(a) Prove that D is a linear operator on P.

(b) Determine the rank and nullity of D as a linear operator on P. Find bases for the nullspace of D and the image of D.

Solution:
It is a linear map: 1) Let $p_1 = \sum_j a_j x^j$, $p_2 = \sum_j b_j x^j$. $D(p_1 + p_2) = \sum_j j(a_j + b_j)x^{j-1}$; $D(p_1) = \sum_j j(a_j)x^{j-1}$, $D(p_2) = \sum_j j(b_j x^{j-1}$.
2) $D(\lambda p_1) = \sum_j j\lambda a_j x^{j-1} = \lambda \sum_j j a_j x^{j-1} = \lambda D(p_1)$.
3) D(0)=0. D is a linear operator as differentiation reduces the total degree by one so $D(p_1) \in V$. From part 1), we know that $D(p) = 0 \to j a_j = 0 \forall j \to$ Null $D = \text{span}\{a_0\}$. So the nullity of D is the dimension of the nullspace(1), and the total dimension is 5 $(a_0 + a_1 x \ldots a_4 x^4)$ so the rank of D (dimension of the Range/image of D)is 5-1=4.
The image of $D(p) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$, $a_i \in \mathbb{R}$. Thus a basis for Im(D)=$\{1, x, x^2, x^3\}$

Problem 2

Let $V = \mathbb{R}^m$, and let W be a subspace of V with basis $\{x_1, x_2, \ldots, x_n\}$. Let v be any vector in V. Derive the normal equations method for computing the best approximation $w \in W$ to v.

Solution:
Let v be a vector in V and let $A = [x_1|x_2|\ldots x_n]$, where the columns of A represent the coordinates of the basis vectors of V with respect to the standard basis for $\mathbb{R}^m$. We can represent any $w = \sum y_j x_j = Ay$, where y is the n dimensional column vector of coordinates of w in the basis $\{x_1, x_2, \ldots, x_n\}$. The best approximation in the least squares sense to v in W minimizes $\|w - v\|$. This is equivalent to minimizing $\|w - v\|^2 = \langle w - v, w - v\rangle = \langle Ay - v, Ay - v\rangle$. Expanding and using the fact that we are using the Euclidean inner product, we have $\langle Ay - v, Ay - v\rangle = (Ay - v)^T(Ay - v) = y^T A^T Ay + v^T v - y^T A^T v - v^T Ay$. As v is fixed, and using the symmetry of the inner product, we may rewrite this minimization as $\min_y f(y) = y^T A^T Ay + v^T v - y^T A^T v - v^T Ay$. As v is fixed, taking the gradient with respect to y and looking for critical points yields: $A^T Ay - A^T v = 0$, so we get $A^T Ay = A^T v$. The Hessian, $D^2 f = A^T A$ is symmetric positive definite (as the columns of A are linearly independent), so any critical point (the solution of the normal equations) is a minimizer.

Problem 3

Let V be a real vector space, let A and B be two subspaces of V, let $\tilde{A}$ be a subspace of V such that $\tilde{A} \oplus (A \cap B) = A$ and let $\tilde{B}$ be a subspace of V such that $\tilde{B} \oplus (A \cap B) = B$. Prove that $A + B = (A \cap B) \oplus \tilde{A} \oplus \tilde{B}$.

Solution: A sum of spaces is a direct sum if and only if its intersection with the sum of the other spaces is empty. Clearly $A + B = (A \cap B) + \tilde{A} + \tilde{B}$. (Should they have to show this, it's fairly trivial?) Checking the 3 intersections: i) $(A \cap B) \cap (\tilde{A} + \tilde{B})$ is zero. Let v be in $A \cap B$ and in $(\tilde{A} + \tilde{B})$. $v = \tilde{a} + \tilde{b}$ by construction, but then $\tilde{a} = v - \tilde{b} \in B$ by subspace closure, but as $\tilde{a} \in A$ this means $\tilde{a} \in A \cap B \to \tilde{a} = 0$. A similar argument yields $\tilde{b} = 0$ so the intersection is zero.
ii) $\tilde{A} \cap (\tilde{B} + (A \cap B)) = B$ is just the zero vector as $\tilde{A} \cap (A \cap B)$ is zero from the direct sum in its definition.
iii) The same argument switching $\tilde{A}$ and $\tilde{B}$.
As all three intersections are empty, the sum of $(A \cap B) + \tilde{A} + \tilde{B}$ is direct.

Problem 4

Let
$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Define
$$\begin{aligned} T : \mathcal{M}_2(\mathbb{R}) &\longrightarrow \mathcal{M}_2(\mathbb{R}) \\ B &\longmapsto AB - BA. \end{aligned}$$

(a) Fix an ordered basis $\mathcal{B}$ of $\mathcal{M}_2(\mathbb{R})$ and compute the matrix $[T]_{\mathcal{B}}$ that represents T with respect to this basis.

(b) Compute a basis for each of the eigenspaces of T.

Solution:

(a) For simplicity, we will let our basis be given by v_{ij}, $v_{ij} = e_i e_j^\top$. We map i, j to a single index k via $k = i + 2(j-1)$. With this choice of basis, we have $T(e_{ij}) = Ae_i e_j^\top - e_i e_j^\top A$. This simplifies to $a_i e_j^\top - e_i a_j^\top$, where we have used that A is symmetric (otherwise we would get jth row of A). The first matrix is equal to
$$\sum_{i=1}^{n} a_{j,i} v_{i,j}.$$
while the second is equal to
$$-\sum_{j=1}^{n} a_{i,j} v_{i,j}$$
yielding
$$\mathcal{M}_T = \begin{pmatrix} 0 & -2 & 2 & 0 \\ -2 & 0 & 0 & 2 \\ 2 & 0 & 0 & -2 \\ 0 & 2 & -2 & 0 \end{pmatrix}$$

(b) Solution using part (a). The characteristic polynomial of $\mathcal{M}_T$ is $z^4 + 16z^2$, and the eigenvalues are thus 0, $4i$, and $-4i$. As the vector space was specified as over the reals, 0 is the only eigenvalue.

Note that $m_1 = -m_4$ and $m_2 = -m_3$, where m_i is the ith column of $\mathcal{M}_T$. Thus the nullspace (eigenspace of eigenvalue 0) of $\mathcal{M}_T$ is two dimensional with basis
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Problem 5

For arbitrary complex scalars a, b, and c, compute the minimal polynomial of the matrix

$$A = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix}.$$

Solution: This is a companion matrix, so the characteristic and minimal polynomials are identical and equal to $-a - bz - cz^2 + z^3$. To show this, the characteristic polynomials is just computed via $\det(A - I\lambda)$. The minimal polynomial, $m(A) = 0$, means that $m(A)v$ must be zero for for all $v \in \mathbb{C}^3$. Taking the standard basis for $\mathbb{C}^3$, we observe that $Ae_1 = e_2$, $A^2e_1 = e_3$. Thus $(a_0I + a_1A + a_2A^2)e_1 = 0$ imples that $a_i = 0$. Therefore the minimal polynomial cannot be quadratic (or lower). The characteristic polynomial is a multiple of the minimal polynomial, and thus they must be equal.

Problem 6

Let $V = \mathbb{P}(\mathbb{R})$, and let U be a subspace of V given by $\text{span}(1, x, x^2)$.

(a) Pick a basis for U, and find the corresponding dual basis.

(b) Given the inner product on $\langle p_1, p_2 \rangle = \int_0^1 p_1(x)p_2(x)\, dx$, find the Riesz representers (in U) of the dual basis in part a). Recall that the Riesz representer is the unique vector u in V s.t., given a fixed ϕ in V', $\phi(v) = \langle v, u \rangle$.

Solution:

(a) We pick the standard basis for $U(u_1, u_2, u_3 = 1, x, x^2)$. The dual basis ϕ_i in U', the dual space of U, are functionals on U s.t. $\phi_i(u_j) = \delta_{ij}$. What is an example of a functional on U s.t. $\phi(1) = 1$ but $\phi(x) = \phi(x^2) = 0$? $\phi_1(p) = p(0)$. The derivative operator, which we encountered in the first problem, works for $\phi_2 = Dp(0)$. Finally, for $\phi_3 = 1/2D^2p(0)$. (In general, the dual basis of a monomial basis will be the corresponding Taylor series coefficient).

(b) The easy way to find the Riesz representors for ϕ_i is to find an orthonormal basis for U and then use the result:

$$u_{\phi_i} = \sum_j \phi_i(o_j)o_j$$

We apply Gram-Schmidt to $\{1, x, x^2\}$.

$o_1 = 1;$

$$o_2 = \frac{x - \langle 1, x \rangle 1}{\|x - \langle 1, x \rangle 1\|} = \frac{(x - 1/2)}{\|x - 1/2\|} = \sqrt{6}(x - 1/2)$$

$$p_3 = x^2 - \langle 1, x^2 \rangle - \langle \sqrt{6}(x - 1/2), x^2 \rangle \sqrt{6}(x - 1/2); \quad o_3 = \frac{p_3}{\|p_3\|}$$

Need to compute o_3, plut in to compute u_{ϕ_i}

There is another way of solving most of the probem. If the choose an orthonormal basis for U(by performing G-S on $\{1, x, x^2\}$ then part b) s trivial-the Riesz representor of the the dual basis is just the orthonormal basis as $\langle o_j, o_i \rangle = \delta_{ij}$. However, it is very difficult to construct the corresponding functional of that representor. If the students followed this approach, stating the definitions of the dual basis but getting stuck, I'd probably award 15 out of 20 points.

Problem 7

Suppose that A and B are two symmetric real $n \times n$ matrices and that A is positive definite. Show that there is an invertible real matrix U such that U^TAU is the identity matrix and U^TBU is diagonal.

Solution:
We will have a two-step approach. First, since A is symmetric positive definite, we will construct W invertible such that $W^TAW = I$. (There is a variety of ways to do this.) Second, once we get such a W, we will look for U invertible such that U^TAU is the identity matrix and U^TBU is diagonal.

$\boxed{\text{Step 1. Method a.}}$ Since A is symmetric positive definite, A has a symmetric positive definite square root B such that $A = B^2$. We note that $B^{-1}AB^{-1} = I$. We set $W = B^{-1}$. So we have $WAW = I$. Since B is symmetric, so is B^{-1}, so is W, so $W = W^T$ and we have $W^TAW = I$ with W is invertible as well.

$\boxed{\text{Step 1. Method b.}}$ Since A is symmetric positive definite, there exists C lower triangular matrix such that $A = CC^T$. (This is called the Cholesky factorization.) C is invertible. So we can write $C^{-1}AC^{-T} = I$. We set $W = C^{-T}$ and we have $W^TAW = I$ with W is invertible.

$\boxed{\text{Step 1. Method c.}}$ Since A is symmetric positive definite, A is diagonalizable in an orthogonormal basis with real positive eigenvalues, so there exists an orthonormal matrix V and a diagonal with positive entries matrix D such that $A = VDV^T$. Now we see that $D^{\frac{1}{2}}V^TAVD^{-\frac{1}{2}} = I$. (The existence of $D^{-\frac{1}{2}}$ is well justified since D is diagonal with positive entries.) (We used the fact that V is orthogonal to use V^T instead of V^{-1} since we have $V^T = V^{-1}$ for orthogonal matrices.) We repeat. We have $D^{-\frac{1}{2}}V^TAVD^{-\frac{1}{2}} = I$. We set $W = VD^{-\frac{1}{2}}$ and we see that we have $W^TAW = I$ with W is invertible.

$\boxed{\text{Step 2.}}$ Let us have a look at W^TBW. This is a symmetric matrix. So we can diagonalize this matrix in an orthonormal basis with real eigenvalues, so there exists an orthogonal matrix Z and a diagonal with real diagonal entries D such that $W^TBW = ZDZ^T$. So we get $(WZ)^TBWZ = D$. We set $U = WZ$ and we have $U^TBU = D$ with U is invertible.

$\boxed{\text{Step 3.}}$ We have that $W^TAW = I$. We multiply on the left by Z^T and on the right by Z, so we get $Z^TW^TAWZ = Z^TZ$. Now we have $U = WZ$ and we have $Z^TZ = I$ so we end up with $U^TBU = I$.

Problem 8

Let V be a finite-dimensional real vector space.

(a) Suppose $T \in \mathcal{L}(V)$ and m is a nonnegative integer such that

$$\text{Range } T^m = \text{Range } T^{m+1}.$$

Prove that Range $T^k =$ Range T^m for all $k > m$.

(b) Prove or give a counterexample: if $T \in \mathcal{L}(V)$, then

$$V = \text{Null } T \oplus \text{Range } T.$$

(c) Prove that if $T \in \mathcal{L}(V)$, then

$$V = \text{Null } T^n \oplus \text{Range } T^n,$$

where $n = \dim V$.

Solution:

(a) Let $k > m$.

We have Range $T^k \subseteq$ Range T^m. (Easy to prove. True for any operator T.)

We want to prove that Range $T^m \subseteq$ Range T^k.

Let x in Range T^m.

Since Range $T^m \subseteq$ RangeT^{m+1}, we have that there exists $y^{(1)}$ in V such that $x = T^{m+1}y^{(1)}$. We rewrite as $x = T(T^m y^{(1)})$.

Since Range $T^m \subseteq$ RangeT^{m+1} and since $T^m y^{(1)}) \in$ Range T^m, we have that there exists $y^{(2)}$ in V such that $T^m y^{(1)} = T^{m+1}y^{(2)}$. We rewrite as $x = T^2(T^m y^{(2)})$.
(This means that Range $T^m \subseteq$ RangeT^{m+2}.)

Since Range $T^m \subseteq$ RangeT^{m+1} and since $T^m y^{(2)}) \in$ Range T^m, we have that there exists $y^{(3)}$ in V such that $T^m y^{(2)} = T^{m+1}y^{(3)}$. We rewrite as $x = T^3(T^m y^{(3)})$.
(This means that Range $T^m \subseteq$ RangeT^{m+3}.)

We continue all the way to getting $x = T^{k-m}(T^m y^{(k-m)}) = T^k y^{(k-m)}$,
which means that Range $T^m \subseteq$ RangeT^k.

(b) We can take the differentiation operator on the polynomial of degree at most 4. (See Problem 1.) The nullspace is $\text{Span}(1)$ and the Range is $\text{Span}(1, x, x^2, x^3)$. So the intersection of the nullspace and the Range of D is not $\{0\}$ and so nullspace and Range are not in direct sum. Another classic example would be the matrix:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

for which we have

$$\text{Range}(A) = \text{Null }(A) = \text{Span}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}),$$

and so nullspace and Range of A are not in direct sum.

(c) $\boxed{\text{Step 1. Preliminary work.}}$
We know that, for all i, Range $T^{i+1} \subseteq$ Range T^i. So we have rank $T^{i+1} \leq$ rank T^i. This says that $(\text{rank } T^i)_{i\in\mathbb{N}}$ is nondecreasing sequence of integers. Clearly $(\text{rank } T^i)_{i\in\mathbb{N}}$ is bounded by n. So $(\text{rank } T^i)_{i\in\mathbb{N}}$ converges to an integer ℓ. ($0 \leq \ell \leq n$.) Say that we "reach" this integer for $i = m$. So we have, for all $i > m$, rank $T^i =$ rank $T^m = \ell$. Now Part (a) tells us that, as soon as $(\text{rank } T^i)_{i\in\mathbb{N}}$ stagnates, it stagnates "forever". This implies that m has to be less than or equal to n. (We have to stagnate before or at n otherwise since we always increase by at least 1 we would end up with a dimension for the rank larger than V, not possible.) So we get

$$\text{for any operator } T, \quad \text{there exists } m \leq n, \quad \text{for all } i > m, \quad \text{rank}T^i = \text{rank}T^m.$$

We can now derive

$$\text{for any operator } T, \quad \text{for all } i > n, \quad \text{rank}T^i = \text{rank}T^n.$$

$\boxed{\text{Step 2.a. Using Range } T^{2n} = \text{Range } T^n \text{ to conclude.}}$
Since for all i, Range $T^{i+1} \subseteq$ Range T^i, we have

$$\text{for any operator } T, \quad \text{for all } i > n, \quad \text{Range } T^i = \text{Range } T^n.$$

In particular, we get

$$\text{for any operator } T, \quad \text{Range } T^{2n} = \text{Range } T^n.$$

Now, let $x \in V$, then $T^n x \in$ Range T^n. But Range $T^{2n} =$ Range T^n, so $T^n x \in$ Range T^{2n}, so there exists y such that $T^n x = T^{2n} y$. Now consider the decomposition

$$x = x - T^n y + T^n y.$$

Clearly $T^n y \in$ Range T^n. And we also have $x - T^n y \in$ Null T^n (since $T^n x = T^{2n} y$). So

$$x \in \text{Range } T^n + \text{Null } T^n.$$

So

$$\text{Range } T^n + \text{Null } T^n = V.$$

Now, since we also have rank $T^n + \dim \text{Null } T^n = n$, we get

$$\text{Range } T^n \oplus \text{Null } T^n = V.$$

$\boxed{\text{Step 2.b. Using Null } T^{2n} = \text{Null } T^n \text{ to conclude.}}$

(Starting from the end of Step 1.) The rank theorem applied to T^i gives rank $T^i + \dim \text{Null } T^i = n$. So we also get

$$\text{for any operator } T, \quad \text{for all } i > n, \quad \dim \text{Null } T^i = \dim \text{Null } T^n.$$

And since for all i, Null $T^i \subseteq$ Null T^{i+1}, we have

$$\text{for any operator } T, \quad \text{for all } i > n, \quad \text{Null } T^i = \text{Null } T^n.$$

In particular, we get

$$\text{for any operator } T, \quad \text{Null } T^{2n} = \text{Null } T^n.$$

Now, let $x \in \text{Null } T^n \cap \text{Range } T^n$, then, since $x \in \text{Range } T^n$, there exists y such that $x = T^n y$. Since $x \in \text{Null } T^n$, $T^n x = 0$, so $T^{2n} y = 0$, so $y \in \text{Null } T^{2n}$. But we know that Null $T^{2n} =$ Null T^n, so $y \in \text{Null } T^n$, $T^n y = 0$, but $x = T^n y$, so $x = 0$.

So

$$\text{Null } T^n \cap \text{Range } T^n = \{0\}.$$

Now, since we also have rank $T^n + \dim \text{Null } T^n = n$, we get

$$\text{Range } T^n \oplus \text{Null } T^n = V.$$