**Part I - solve all 4 problems**

1. Let $\{y_n\}$ be a sequence of real numbers. Prove that there exists a continuous function $f : \Re \to \Re$ with $f(\frac{1}{n}) = y_n$ if and only if the sequence $\{y_n\}$ converges.

   Proof: (=>) Suppose $f(\frac{1}{n}) = y_n$. Since $f$ is continuous and $\lim_{n\to\infty} \frac{1}{n} = 0$, it follows that $\lim_{n\to\infty} y_n = f(0)$.

   (<=) Suppose $\lim_{n\to\infty} y_n = y_0$. Construct $f(x)$ as a piecewise linear interpolation between the values
$$f\left(\frac{1}{n}\right) = y_n, \quad n = 1, 2, \ldots$$
   extended by constants outside of the interval $(0, 1)$,
$$f(x) = \begin{cases} y_0 & \text{if } x \le 0 \\ y_1 & \text{if } x \ge 1 \\ y_{n+1} + (x - \frac{1}{n+1})\frac{y_n - y_{n+1}}{\frac{1}{n} - \frac{1}{n+1}} & \text{if } x \in [\frac{1}{n+1}, \frac{1}{n}],\ n = 1, 2, \ldots \end{cases}$$
   Function $f$ is therefore continuous except possibly at $x = 0$. To show that $f$ is continuous at $x = 0$, let $\epsilon > 0$. Since $\lim_{n\to\infty} y_n = y_0$, there exists $N$ such that $|y_n - y_0| < \epsilon$ if $n \ge N$. Let $x$ be such that $|x| < \frac{1}{N}$. If $x \le 0$, then $f(x) = y_0$ by the definition of $f$. If $x > 0$, then $x \in [\frac{1}{n+1}, \frac{1}{n}]$ for some $n \ge N$. Since $f$ is linear on $[\frac{1}{n+1}, \frac{1}{n}]$, so is $f - y_0$, and using the definition of $f$,
$$|f(x) - y_0| \le \max\{|y_{n+1} - y_0|, |y_n - y_0|\} < \epsilon.$$

2. (a) Prove that if $(c_n)$ is an increasing bounded sequence, then it converges.

   (b) Let
$$c_n = \sum_{i=1}^{n} \frac{1}{n+i}$$
   Prove that $(c_n)$ converges. Hint: Use part (a).

   Proof: (a) Since $(c_n)$ is bounded, the set $\{c_n\}$ has a supremum $c \in \Re$. Let $\epsilon > 0$. Since $c$ is upper bound on $\{c_n\}$, we have $c_n \le c < c + \epsilon$ for all $n$. Since is the least upper bound, $c - \epsilon$ is not an upper bound, and there exists $N$ such that $c_N > c - \epsilon$. Since $\{c_n\}$ is increasing, $c_n \ge c_N > c - \epsilon$ for all $n \ge N$. Thus, $|c - c_n| < \epsilon$ for all $n \ge N$. Since $\epsilon > 0$ was arbitrary, $c_n \to c$ as $n \to \infty$.

(b) We have

$$\begin{aligned}
c_{n+1} - c_n &= \sum_{i=1}^{n+1} \frac{1}{n+1+i} - \sum_{i=1}^{n} \frac{1}{n+i} \\
&= \left( \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2} \right) \\
&- \left( \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) \\
&= \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1} > \frac{1}{2n+2} + \frac{1}{2n+2} - \frac{1}{n+1} = 0,
\end{aligned}$$

so $(c_n)$ is increasing. Also, $(c_n)$ is bounded since

$$0 < c_n = \sum_{i=1}^{n} \frac{1}{n+i} < \sum_{i=1}^{n} \frac{1}{n} = 1.$$

Since the sequence $(c_n)$ is increasing and bounded, it converges by part (a).

3. Prove that $\Re$ is not compact, using (a) open covers, and (b) sequences..

   Proof (a) Let $O_n = (n-1, n+1), n \in Z$. Then $\{O_n\}$ is an open cover of $\Re$. Let $\{O_{n_1}, \ldots, O_{n_k}\}$ be a finite subset of $\{O_n\}$. Define $m = \max(|n_1|, ..., |n_k|)$. Then $\cup_{i=1}^{k} O_{n_i} \subset (-(m+1), m+1) \neq \Re$. Thus, $\{O_n\}$ has no finite subcover, so $\Re$ is not compact.

   (b) Let $x_n = n,\ n = 1, 2, ....$ Then every subsequence of $\{x_n\}$ diverges to $\infty$, so $\Re$ is not compact.

4. Suppose $f_n : \Re \to \Re$ is bounded for each $n = 1, 2, ...$, i.e., $|f_n(x)| \leq c_n$, Suppose $f_n \to f$ uniformly. Prove or find a counterexample: $f$ is bounded.

   Proof: Since $f_n \to f$ uniformly, there exists $n$ such that

   $$\forall x \in \Re : |f_n(x) - f(x)| < 1.$$

   Let $x \in \Re$. By triangle inequality,

   $$|f(x)| \leq |f_n(x)| + |f(x) - f_n(x)| < c_n + 1,$$

   so $f$ is bounded.

**Part 2 - choose 2 out of 4**

5. (a) Suppose $x > -1$. Use the Taylor's Theorem to express $f(x) = \ln(1+x)$ with the remainder in the $x^3$ term.

   (b) Determine
$$\lim_{n\to\infty} e^{-nx}(1+\frac{x}{n})^{n^2}$$
   for any $x \in \Re$. Hint: Take the logarithm and use part (a).

Proof: (a) Suppose $x > -1$. Then,

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1}{(1+x)^2}, \quad f'''(x) = \frac{2}{(1+x)^3}.$$

Since $f'''$exists and $f''$ is continous on $(1,\infty)$, from Taylor's theorem, there exists $\xi_x$ between 0 and $x$ such that

$$f(x) = f(0) + f'(x)x + \frac{1}{2}f''(x)x^2 + \frac{1}{6}f'''(\xi_x)x^3,$$

that is,

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}\frac{1}{(1+\xi_x)^3}x^3.$$

(b) Denote $y_n(x) = e^{-nx}(1+\frac{x}{n})^{n^2}$. Let $x \in \Re$. Then from (a), for sufficiently large $n$ so that $\frac{x}{n} > -1$,

$$\begin{aligned}
\ln y_n(x) &= -nx + n^2\ln(1+\frac{x}{n}) \\
&= -nx + n^2\left(\frac{x}{n} - \frac{1}{2}\left(\frac{x}{n}\right)^2 + \frac{1}{3}\frac{1}{\left(1+\xi_{\frac{x}{n}}\right)^3}\left(\frac{x}{n}\right)^3\right) \\
&= -\frac{x^2}{2} + \frac{x^2}{3n\left(1+\xi_{\frac{x}{n}}\right)^3} \to -\frac{x^2}{2} \text{ as } n\to\infty
\end{aligned}$$

since $\xi_{\frac{x}{n}}$ is between 0 and $\frac{x}{n}$, thus $\xi_{\frac{x}{n}} \to 0$ as $n\to\infty$. So, $\lim_{n\to\infty} y_n(x) = e^{-x^2/2}$.

6. (a) Write the power series for $\frac{1}{1-x}$, prove that it converges on $(-1,1)$, and that the convergence is uniform on any interval $[-r,r]$ with $0 < r < 1$.

   (b) Integrate the power series for $\frac{1}{1+x}$to get the power series for $\ln(1+x)$ and justify your steps.

   (c) Find $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ from (b) and justify your steps.

Proof: (a) Let $x \in (-1,1)$ .Denote $S = \sum\limits_{n=0}^{\infty} x^n$. The partial sums $S_n = 1+x+\cdots+x^n \to$ .$S$ as $n\to\infty$ by the root test. Since $S_n - xS_n = 1 - x^{n+1}$, we have

$$S_n = \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x} \to \frac{1}{1-x} \text{ as } n\to\infty$$

and from uniqueness of limit, $S = \frac{1}{1+x}$. Let $0 < r < 1$. Then, for all $x \in [-r, r]$ and all $n$, $|x|^n \le r^n$. Since $\sum\limits_{n=0}^{\infty} r^n$ converges by the root test, $\sum\limits_{n=0}^{\infty} x^n$ converges uniformly on $[-r, r]$ by the Weierstrass $M$ test.

(b) Corollary to Theorem 7.16 says that if a series of functions converges uniformly on $[a, b]$ then it can be integrated $[a, b]$ term by term. Let $x \in (-1, 1)$, and use (a) with $-u$ for $x$ and $|x|$ for $r$. Then, the series $\sum_{n=0}^{\infty}(-u)^n$ converges uniformly on $[-r, r]$, $r = |x|$ , and we can integrate term by term on $[0, x]$ if $x > 0$ and on $[x, 0]$ if $x < 0$,

$$\ln(1+x) = \int_0^x \frac{du}{1+u} = \int_0^x \sum_{n=0}^{\infty}(-u)^n du = \sum_{n=0}^{\infty} \int_0^x (-u)^n du = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}.$$

(c) From (b) we know that the power series for $\ln(1 + x)$ converges for $|x| < 1$. Since $\frac{1}{n}$ is a decreasing sequence, the alternating series theorem says $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. So by theorem 8.2 (Abel's thm), we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \lim_{x \to 1} \ln(1+x) = \ln 2.$$

7. Let $\hat{\mathcal{C}}[0, 1]$ be the set of continuous function, $f : [0, 1] \to [0, 1]$ For $f, g \in \hat{\mathcal{C}}[0, 1]$, let

$$d_1(f, g) = \int_0^1 |f(x) - g(x)|dx$$

$$d_2(f, g) = \sup_{0 \le x \le 1} |f(x) - g(x)|.$$

Let $X$ be the set of sequences $(x_1, x_2, \ldots)$ where each $x_i \in [0, 1]$, and for $x, y \in X$, let

$$\hat{d}(x, y) = \sup_i |x_i - y_i|$$

Let $F : \hat{\mathcal{C}}[0, 1] \to X$ be given by

$$F(f) = (f(1), f(1/2), f(1/3), \ldots)$$

(a) Verify that $d_1, d_2, \hat{d}$ are metrics.

(b) For $i = 1, 2$ prove or find a counterexample: $F : (\hat{\mathcal{C}}[0, 1], d_i) \to (X, \hat{d})$ is continuous

Proof: $d_1 \to \hat{d}$ is not continuous. Counterexample: Define $f_n$ to be 0 except on $\left(\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}\right)$, where $f_n$ is piecewise linear given by the values

$$f\left(\frac{1}{2} - \frac{1}{n}\right) = 0, \quad f\left(\frac{1}{2}\right) = 1, \quad f\left(\frac{1}{2} + \frac{1}{n}\right) = 0.$$

Then $d_1(f_n, 0) = \int_0^1 f_n = 1/n \to 0$ as $n \to \infty$, but $f_n(1/2) = 1$ for every $n$, so for every $n$, $\hat{d}(F(f_n), F(0)) = 1$.

$d_2 \to \hat{d}$ is continuous since $\sup_{0 \le x \le `} |f(x) - g(x)| < \epsilon \Rightarrow \sup_k |f(1/k) - g(1/k)| < \epsilon$.

8. (a) Prove or find a counterexample: If $f$ is continuous on $[a, b]$ then for any $\epsilon > 0$ there is a polynomial $p(x)$ such that $\int_a^b |f(x) - p(x)|dx < \epsilon$.

(b) Prove or find a counterexample: If $f$ is Riemann integrable on $[a, b]$ then for any $\epsilon > 0$ there is a polynomial $p(x)$ such that $\int_a^b |f(x) - p(x)|dx < \epsilon$.

Proof: (a) Let $\epsilon > 0$. From the Weierstrass theorem, there exists polynomial $p(x)$ such that $\sup_{x\in[a,b]} |f(x) - p(x)| < \epsilon/(b - a)$, so

$$\int_a^b |f(x) - p(x)|dx < \int_a^b \epsilon/(b - a)dx = \epsilon$$

(b) Let $\epsilon > 0$. Since $f$ is Riemann integrable on $[a, b]$ there is a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ such that

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} \left( \sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \right) (x_i - x_{i-1}) < \frac{\epsilon}{3}.$$

Define non-overlapping intervals $I_1 = [x_0, x_1]$, $I_i = (x_{i-1}, x_i]$, $i = 2, \ldots .n$ and piecewise constant function $g$ by

$$g(x) = \sup_{[x_{i-1},x_i]} f \quad \text{if } x \in I_i.$$

Then, $\int_a^b |f(x) - g(x)|dx < \epsilon/3$ since

$$|f(x) - g(x)| \le \sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \text{ if } x \in [x_{i-1}, x_i]$$

thus

$$\int_a^b |f(x) - g(x)|dx \le \sum_{i=1}^{n} \left( \sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \right) (x_i - x_{i-1}) < \frac{\epsilon}{3}.$$

Construct $h(x)$ from $g(x)$ by replacing the jumps by linear interpolations so that $h$ is continuous and

$$\int_a^b |g(x) - h(x)|dx < \epsilon/3.$$

E.g., choose $\delta = \min\left\{ \frac{\epsilon}{6M(n-1))}, \frac{x_1 - x_0}{2}, \ldots, \frac{x_{n-1} - x_n}{2} \right\}$. Note that $x_i + \delta < x_{i+1} - \delta$ and define $h$ piecewise linear with the same values as $g$ at the points

$$x_0 = a, x_1 - \delta, x_1 + \delta, x_2 - \delta, \ldots, x_{n-1} + \delta, x_{n-1} + \delta, x_n = b$$

Now choose polynomial $p(x)$ so that $\int_a^b |h(x) - p(x)|dx < \epsilon/3$ (which is possible from part (a)). Then

$$\int_a^b |f - p| \le \int_a^b |f - g| + \int_a^b |g - h| + \int_a^b |h - p| < \epsilon.$$