Applied Analysis Preliminary Examination—January 2016

Name:

- Turn in problems 1 to 4, and exactly two out of problems 5,6,7. Only 6 solutions will be graded. Each problem is worth 20 points.
- Be sure to show all your relevant work. Rewrite your solutions, if necessary, so they are neat and easy to read.
- **Only write on one side of each sheet.**
- Write your name and the problem number on every sheet. Number the pages within each problem. If you do not use the page with the statement of the problem, copy the problem statement before the solution.
- Justify your solutions.
- If you use a theorem from Rudin or class, state it. If you are unsure if a statement must be proved or may merely be stated, ask your friendly proctor.

1	2	3	4	5	6	7	$\sum$

Section 1: Complete ALL four of the following questions.

1. Let (a_n) and (b_n) be bounded nonnegative sequences. Prove that

$$\limsup_{n\to\infty} a_n b_n \leq \left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right).$$

2. Suppose that (M, d) is a compact metric space. Prove that for every $\varepsilon > 0$, there exists a finite set $A \subset M$ such that the distance of every point M to A is less than ε.

3. Let f and g be continuous maps of a metric space (X, d_X) into a metric space (Y, d_Y). Let $h : X \to \mathbb{R}$ be defined by $h(x) = d_Y(f(x), g(x))$. Prove h is continuous and that the set $\{x \in X : f(x) = g(x)\}$ is closed.

4. Let $(\mathcal{C}([a, b]), d)$ denote the metric space of continuous functions on $[a, b]$, where $a < b$ are real numbers, and $d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$. Let $(f_n) \subset \mathcal{C}([a, b])$ be a uniformly equicontinuous sequence of functions that converge pointwise to f on $[a, b]$. Prove that f is continuous on $[a, b]$.

Section 2: Complete exactly TWO of the following three questions. If you submit three problems, only the first two will be graded.

5. Prove that there exists exactly one $x \in [1, +\infty)$ such that $x = 1 + \sin \frac{x}{2}$, using the contraction theorem (also known as the Banach contraction principle). Verify all assumptions of the theorem.

6. Let (g_k) be a sequence of real-valued functions defined on $S \subset \mathbb{R}$. If $\sum_{k=1}^{\infty} g_k$ converges uniformly on S to real-valued function g, prove that $g_k \to 0$ uniformly on S as $k \to \infty$.

7. Prove the following theorem.

   Suppose (f_n) is a sequence of real-valued continuous functions on the interval $[a, b]$, $a < b$, and the derivatives f_n' are continuous on $[a, b]$. If

   (a) the sequence of derivatives (f_n') converges uniformly on $[a, b]$ to $g : [a, b] \to \mathbb{R}$, and

   (b) there exists a point $x_0 \in [a, b]$ such that $\lim_{n\to\infty} f_n(x_0)$ exists,

   then the functions f_n converge uniformly to a differentiable function f on $[a, b]$ such that $f' = g$ on $[a, b]$.