PHD PRELIMINARY EXAMINATION IN APPLIED ANALYSIS
JUNE 1, 2018

Name: ______________________________

- The examination consists of 6 problems out of 7. If you submit all 7, 1 to 6 will be graded.
- Each problem is worth 20 points. Unless specified otherwise, numbered parts of a problem have equal weight.
- Justify your solutions: cite theorems that you use, provide counter-examples, give explanations.
- Write legibly using a dark pencil or pen. Rewrite your solution if it gets too messy.
- Please begin solution to every problem on a new page; **write only on one side of paper**; number all pages throughout; and, just in case, write your name on every page.
- Do not submit scratch paper or multiple alternative solutions. If you do, we will grade the first solution to its end and we will not attempt to fish for the truth.
- Ask the proctor if you have any questions.

Good luck!

1. __________

2. __________

3. __________

4. __________

5. __________

6. __________

7. __________

Total _______

Examination committee: Troy Butler, Jan Mandel (chair), Florian Pfender.

(1) Decide if functions $f_n(x) = e^{-\left|x-\frac{1}{n}\right|n^2}$ (a) converge on $\mathbb{R}$ pointwise, (b) converge on $\mathbb{R}$ uniformly.

(2) Decide if the function $f\left(x,y\right)=\frac{\sin xy}{x^{2}+y^{2}}$ can be continously extended to all of $\mathbb{R}^{2}$.

(3) Let (X, d) be a metric space.

(a) Prove that d is a continuous real-valued function on the product metric space $(X \times X, d_{X\times X})$ where $d_{X\times X}$ is a natural product metric induced by d.

(b) Give an example of (X, d) and complete non-empty subsets $A, B \subset X$ such that there do not exist $a_0 \in A$ and $b_0 \in B$ such that

$$d(a_0, b_0) = \inf\{d(a, b) \,:\, a \in A, b \in B\}$$

(4) We say that two metrics d_1 and d_2 defined on the same space X are equivalent if there exists real numbers $c_1 > 0$ and $c_2 > 0$ such that for every $x, y \in X$,

$$c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y).$$

(a) Prove that if d_1 and d_2 are equivalent metrics, then a sequence $(x_n) \subset X$ converges to x in (X, d_2) if and only if $(x_n) \subset X$ converges to x in (X, d_1).

(b) Let $C([0, 1])$ denote the space of all continuous real-valued functions on $[0, 1]$. For any $f, g \in C([0, 1])$, let d_I denote the *integral* metric defined by

$$d_I(f, g) = \int_0^1 |f(x) - g(x)|\, dx,$$

and d_S denote the *supremum* metric defined by

$$d_S(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Prove that the metrics d_I and d_S are *not* equivalent..

(5) Let $\mathcal{F}$ be a bounded subset of $C([a,b])$ with the supremum metric and

$$A = \left\{ F(x) = \int_a^x f(t)\,dt \;:\; f \in \mathcal{F} \right\}.$$

Prove that the closure $\overline{A}$ of A is a compact subset of $C([a,b])$.

(6) Let (X, d) be a complete metric space, and $A \subset X$, equiped with the distance function d restricted to $A \times A$, denoted by d_A. Prove that the space (A, d_A) is complete if and only if A is closed in (X, d).

(7) Consider the power series $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$

(a) (5 points) Decide for which real numbers x the series converges.

(b) (15 points) Decide on which intervals the series converges uniformly.