MATH 5070 PRELIMINARY EXAM FOR SUMMER 2017
SOLUTIONS

(1) Let (X, d) be a metric space, $K \subset X$ be nonempty, and let K' denote the set of limit points of K. Define the closure of K as $\overline{K} := K \cup K'$. Prove that $\overline{K}$ is both (1) closed, and (2) if F is closed and $K \subset F$, then $\overline{K} \subset F$. In other words, prove that $\overline{K}$ is the smallest closed set containing K.

Proof. We prove $\overline{K}$ is closed by showing $\overline{K}^c$ is open. Consider any $x \in \overline{K}^c$, then x is neither in K nor a limit point of K, so there exists a neighborhood around x that does not intersect K. This implies that $\overline{K}^c$ is open, which proves (1).

If F is closed then F contains all of its limit points, so if $K \subset F$, then by definition $K' \subset F$, which implies $\overline{K} \subset F$. $\square$

(2) Let (X, d) be a metric space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Prove that if there exists $x \in X$ such that for every subsequence $(x_{n_k})_{k\in\mathbb{N}}$ there exists a subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}}$ such that $x_{n_{k_j}} \to x$, then $x_n \to x$.

Proof. Suppose that every subsequence $(x_{n_k})_{k\in\mathbb{N}}$ has a subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}}$ such that $x_{n_{k_j}} \to x$, but assume to the contrary that $x_n \not\to x$. Then, there exists an $\epsilon > 0$ such that for every $N \in \mathbb{N}$ there exists $n > N$ such that $d(x_n, x) \geq \epsilon$. Choose such an ϵ and construct a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $d(x_{n_k}, x) \geq \epsilon$ inductively as follows. For $k = 1$, let n_1 be the first index of the sequence such that $d(x_{n_1}, x) \geq \epsilon$. For $k = 2$, let $n_2 > n_1$ be an index such that $d(x_{n_2}, x) \geq \epsilon$. Having chosen the first k terms in the subsequence, choose n_{k+1} such that $n_{k+1} > n_k$ and $d(x_{n_{k+1}}, x) \geq \epsilon$. By construction, any subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}}$ of $(x_{n_k})_{k\in\mathbb{N}}$ will also have the property that $d(x_{n_{k_j}}, x) \geq \epsilon$, which contradicts the hypothesis. □

(3) Let (X, d_X) and (Y, d_Y) be metric spaces, $K \subset X$ nonempty and open, and $f : K \to Y$. Let $\overline{K}$ denote the closure of K (see problem 1 for definition). Suppose Y is complete and f is uniformly continuous.

(a) **(15 points)** Prove that there exists a unique uniformly continuous function $\overline{f} : \overline{K} \to Y$ such that $\overline{f}(x) = f(x)$ for every $x \in K$. We call $\overline{f}$ the extension of f to $\overline{K}$.

(b) **(5 points)** Give (1) an example showing the necessity of the condition that Y is complete, and (2) an example showing that even if Y is complete but f is only continuous, then there may not be an extension of f to $\overline{K}$ that is continuous.

Part (a)

Proof. Consider any $x \in \overline{K}$, then there exists $(x_n) \subset K$ such that $x_n \to x$, which implies (x_n) is Cauchy. Since f is uniformly continuous, $(f(x_n))$ is Cauchy by a standard result. Since Y is complete, $(f(x_n))$ converges to some number that we define to be $\overline{f}(x)$.

This way of defining $\overline{f}$ is both well-defined and unique since if $x_n \to x$ and $z_n \to x$, then we can define (u_n) so that every even term defines the subsequence given by (x_n) and every odd term defines the subsequence given by (z_n), which is Cauchy by construction (this is easily proven by an $\epsilon/2$ argument). Since (u_n) is Cauchy with convergent subsequences, it converges by a standard result, and the uniqueness of limits immediately gives $f(z_n) \to \overline{f}(x)$.

We now show that $\overline{f}$ is uniformly continuous on $\overline{K}$.

Let $\epsilon > 0$.

Since f is uniformly continuous on K, there exists $\delta > 0$ such that $d_Y(f(x), f(z)) < \epsilon/3$ for any $x, z \in K$ with $d_X(x, z) < \delta$. Choose such a δ.

Let $x, z \in \overline{K}$ such that $d_X(x, z) < \delta/3$, and choose $(x_n) \subset K$ and $(z_n) \subset K$ such that $x_n \to x$ and $z_n \to z$. This implies that $f(x_n) \to \overline{f}(x)$ and $f(z_n) \to \overline{f}(z)$. There exists N_1, N_2, N_3, and N_4 such that $n \geq N_1$, $n \geq N_2$, $n \geq N_3$, and $n \geq N_4$ implies $d_X(x, x_n) < \delta/3$, $d_X(z, z_n) < \delta/3$, $d_Y(\overline{f}(x), f(x_n)) < \epsilon/3$, and $d_Y(f(z_n), \overline{f}(z)) < \epsilon/3$, respectively. Choose $n \geq \max\{N_1, N_2, N_3, N_4\}$. For such an n, by repeated use of the triangle inequality,

$$d_X(x_n, z_n) \leq d_X(x_n, x) + d_X(x, z) + d_X(z, z_n) < \delta/3 + \delta/3 + \delta/3 = \delta,$$

which implies that

$$d_Y(f(x_n), f(z_n)) < \epsilon/3.$$

Therefore, by repeated use of the triangle inequality,

$$d_Y(\overline{f}(x), \overline{f}(z)) \leq d_Y(\overline{f}(x), f(x_n)) + d_Y(f(x_n), f(z_n)) + d_Y(f(z_n), \overline{f}(z)) < \epsilon.$$

□

Part (b)

Two examples are required.

For the first example showing the necessity of Y being complete, suppose $Y = (0, 1)$, $X = \mathbb{R}$, and consider $K = (0, 1) \subset X$ with $f(x) = x$. There is no way to define $f(0)$ and $f(1)$ since any sequence $(x_n) \subset K$ that converges to either 0 or 1 is Cauchy, but not convergent, in Y.

For the second example, take $Y = \mathbb{R}$, $X = \mathbb{R}$, $K = (0, 1] \subset X$, and $f(x) = 1/x$, which is easily seen to not have any continuous extension at $x = 0$ since the limit of $f(x)$ as x approaches 0 within K is $+\infty$.

(4) Let (X, d_X) and (Y, d_Y) be metric spaces, X compact, and $f : X \to Y$ satisfies two conditions

(i) For each compact set $K \subset X$, $f(K)$ is compact.

(ii) For every nested decreasing sequence of compact sets $(K_n) \subset X$,

$$f(\cap K_n) = \cap f(K_n).$$

Prove that f is continuous.

Proof. We prove by contradiction.

Assume that f is not continuous.

Then, there exists an $x \in X$ and $\epsilon > 0$ such that for each $n \in \mathbb{N}$, there exists $x_n \in B_{1/n}(x)$ such that $f(x_n) \notin B_\epsilon(f(x))$.

For each $n \in \mathbb{N}$, let $K_n = \overline{B}_{1/n}(x)$ denote the closure of the ball $B_{1/n}(x)$. Since closed subsets of a compact space are compact by a standard result, K_n is compact for each $n \in \mathbb{N}$. By construction, (K_n) is a nested sequence of compact sets and $\cap K_n = \{x\}$.

By assumption (i), $f(K_n)$ is compact for each $n \in \mathbb{N}$ and $(f(K_n))$ is a nested decreasing sequence of compact sets in Y by construction. Since $(f(x_n)) \subset f(K_1)$, there exists a convergent subsequence $(f(x_{n_k}))$. By construction, $(f(x_{n_k}))_{k \geq N} \subset f(K_N)$ for each $N \in \mathbb{N}$, and since $f(K_N)$ are closed (since they are compact) for each $N \in \mathbb{N}$, the limit of $(f(x_{n_k}))$ belongs to $\cap f(K_n)$. By assumption (ii) on f, $\cap f(K_n) = f(\cap K_n) = \{f(x)\}$, which implies $f(x_{n_k}) \to f(x)$ contradicting how $(f(x_n))$ was constructed. □

(5) Suppose $f : [-1,1] \to \mathbb{R}$ is three-times differentiable with continuous third derivative on $[-1,1]$. Prove that the series

$$\sum_{n=1}^{\infty} \left[n\big(f(1/n) - f(-1/n)\big) - 2f'(0)\right]$$

converges.

Proof. By Taylor's theorem, for each $n \in \mathbb{N}$, there exists $\xi_n^{(1)} \in (0, 1/n)$ such that

$$f(1/n) = f(0) + f'(0)\frac{1}{n} + f''(0)\frac{1}{2n^2} + f'''(\xi_n^{(1)})\frac{1}{6n^3},$$

and there exists $\xi_n^{(2)} \in (-1/n, 0)$ such that

$$f(-1/n) = f(0) - f'(0)\frac{1}{n} + f''(0)\frac{1}{2n^2} - f'''(\xi_n^{(2)})\frac{1}{6n^3}.$$

Then, we have that for each $n \in \mathbb{N}$, we see that

$$\left[n\big(f(1/n) - f(-1/n)\big) - 2f'(0)\right] = \frac{1}{6n^2}\left[f'''(\xi_n^{(1)}) + f'''(\xi_n^{(2)})\right].$$

Then, since the third derivative is continuous on $[-1,1]$, it is bounded in magnitude on $[-1,1]$ by some $M \geq 0$, so that

$$\frac{1}{6n^2}\left|f'''(\xi_n^{(1)}) + f'''(\xi_n^{(2)})\right| \leq \frac{M}{3n^2}.$$

Since

$$\sum_{n=1}^{\infty} \frac{M}{3n^2}$$

converges by the integral test, we have that the series converges (in fact converges absolutely). $\square$

(6) Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. Prove that f is uniformly continuous if and only if for every sequences $(x_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ in X such that $d_X(x_n, z_n) \to 0$ implies $d_Y(f(x_n), f(z_n)) \to 0$.

Proof. First assume that f is uniformly continuous. Let $\epsilon > 0$. There exits $\delta > 0$ such that $d_X(x, z) < \delta$ implies $d_Y(f(x), f(z)) < \epsilon$. Choose such a $\delta > 0$. Consider any sequences $(x_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ in X such that $d_X(x_n, z_n) \to 0$. Then, there exists N such that $n \geq N$ implies $d_X(x_n, z_n) < \delta$, which implies that $d_Y(f(x_n), f(z_n)) < \epsilon$. Thus, $d_Y(f(x_n), f(z_n)) \to 0$.

Now assume that f is not uniformly continuous. Then, there exists $\epsilon > 0$ such that for every $\delta > 0$ there exists $x, z \in X$ with $d_X(x, z) < \delta$ and $d_Y(f(x), f(y)) \geq \epsilon$. Choose such an ϵ, and for each $n \in \mathbb{N}$ let $\delta_n = 1/n$, and choose $x_n, z_n \in X$ such that $d_X(x_n, z_n) < \delta_n$ and $d_Y(f(x_n), f(y_n)) \geq \epsilon$. By construction, $d_X(x_n, z_n) \to 0$ but $d_Y(f(x_n), f(z_n)) \not\to 0$. $\square$

(7) Let $f : [0,1] \to \mathbb{R}$ be continuously differentiable with $f(0) = 0$. Prove that

$$[\sup\{|f(x)| \,:\, 0 \le x \le 1\}]^2 \le \int_0^1 (f'(x))^2\,dx.$$

Proof. By the Fundamental Theorem of Calculus (and the fact that $f(0) = 0$), for each $x \in [0,1]$,

$$f(x) = \int_0^x f'(s)\,ds \Rightarrow |f(x)| \le \int_0^x |f'(s)|\,ds.$$

By the standard Cauchy-Schwartz (or just Schwartz) inequality

$$\begin{aligned}\int_0^x |f'(s)|\,ds &\le \left(\int_0^x |f'(s)|^2\,ds\right)^{1/2}\left(\int_0^x 1^2\,ds\right)^{1/2}\\ &\le \left(\int_0^1 |f'(x)|^2\,dx\right)^{1/2}.\end{aligned}$$

Thus, for each $x \in [0,1]$,

$$|f(x)| \le \left(\int_0^1 |f'(x)|^2\,dx\right)^{1/2}.$$

Since the inequality holds for all $x \in [0,1]$,

$$\sup\{|f(x)| \,:\, 0 \le x \le 1\} \le \left(\int_0^1 |f'(x)|^2\,dx\right)^{1/2}.$$

Squaring both sides completes the proof. □