1. Let (X, d) be a metric space.

   (a) Prove or find a counterexample: If (x_n) is a Cauchy sequence in (X, d), then (x_n) converges.

   (b) Prove that if (x_n) and (y_n) are both Cauchy sequences in (X, d), then the sequence $(d(x_n, y_n))$ converges.

Solution.

(a) We use the rationals $\mathbb{Q}$, equipped with the stardard metric. We use known facts about rationals: $\mathbb{Q}$ is dense in $\mathbb{R}$, and $\sqrt{2} \notin \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a sequence $(x_n) \subset \mathbb{Q}$, $x_n \to \sqrt{2}$ in $\mathbb{R}$. Since (x_n) converges in $\mathbb{R}$, (x_n) is Cauchy in $\mathbb{R}$. Since the $\mathbb{Q}$ is a subspace of $\mathbb{R}$, the metric is the same, and thus (x_n) is Cauchy in $\mathbb{Q}$. Now if $x_n \to x$ in $\mathbb{Q}$, then also $x_n \to x$ in $\mathbb{R}$ and by uniqueness of limit, $x = \sqrt{2} \notin \mathbb{Q}$, contradiction.

Note: Saying that "the limit is outside of the space" is not sufficient. Limit in what metric space? Correct solution along those lines needs to involve two spaces and a uniqueness of limit argument.

(b) We use the inequality

$$|d(x_n, y_n) - d(x_m, y_m)| \le d(x_n, x_m) + d(y_n, y_m)$$

to show that $(d(x_n, y_n))$ is Cauchy in $\mathbb{R}$: Let $\varepsilon > 0$. Since (x_n) is Cauchy in (X, d), there exists N_1 such that for all $m, n \ge N_1$, $d(x_n, x_m) < \frac{\varepsilon}{2}$. Since (y_n) is Cauchy in (X, d), there exists N_2 such that for all $m, n \ge N_2$, $d(y_n, y_m) < \frac{\varepsilon}{2}$. Then,

$$|d(x_n, y_n) - d(x_m, y_m)| \le d(x_n, x_m) + d(y_n, y_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\mathbb{R}$ is complete and $(d(x_n, y_n))$ is Cauchy, $(d(x_n, y_n))$ converges.

2. Let (a_n), (b_n) be sequences in $\mathbb{R}$ and $\lim_{n\to\infty} a_n = a \in \mathbb{R}$.

   (a) Prove that if $a > 0$, then
$$\liminf_{n\to\infty} a_n b_n = a \liminf_{n\to\infty} b_n.$$

   (b) Provide a counterexample when the statement fails with $a = 0$ and $\liminf_{n\to\infty} b_n \in \mathbb{R}$.

Solution.

(a) By definition, for any sequence, $\liminf_{n\to\infty} x_n$ is the infimum of subsequential limits $\lim_{k\to\infty} x_{n_k}$, and it is known that a subsequence exists such that $\lim_{k\to\infty} x_{n_k} = \liminf_{n\to\infty} x_n$.
Denote
$$b = \liminf_{n\to\infty} b_n.$$
Then there exists subsequence (b_{n_k}) such that $\lim_{k\to\infty} b_{n_k} = b$. Since $\lim_{n\to\infty} a_n = a$, we have $\lim_{k\to\infty} a_{n_k} = a$, and since $a \neq 0$, $\lim_{k\to\infty} a_{n_k} b_{n_k} = ab$ (regardless if b is finite or not). Since $a > 0$,
$$\liminf_{n\to\infty} a_n b_n \leq ab.$$
In the opposite direction, suppose that $(a_{n_k} b_{n_k})$ is any convergent subsequence of $(a_n b_n)$. Since $\lim_{n\to\infty} a_n$, the subsequence (a_{n_k}) has the same limit $\lim_{k\to\infty} a_{n_k} = a$, and since $a \neq 0$,
$$\lim_{k\to\infty} b_{n_k} = \lim_{k\to\infty} \frac{a_{n_k} b_{n_k}}{a_{n_k}} = \frac{\lim_{k\to\infty} a_{n_k} b_{n_k}}{\lim_{k\to\infty} a_{n_k}} = \frac{\lim_{k\to\infty} a_{n_k} b_{n_k}}{a}$$
thus
$$\lim_{k\to\infty} a_{n_k} b_{n_k} = a \lim_{k\to\infty} b_{n_k}$$
Since $a > 0$ and $\lim_{k\to\infty} b_{n_k} \geq b$,
$$a \lim_{k\to\infty} b_{n_k} \geq ab$$
Thus,
$$\liminf_{n\to\infty} a_n b_n \geq ab.$$
Note: Since boundedness of (b_n) was not assumed, we need to be careful to use an argument that works also when $\liminf_{n\to\infty} b_n = \infty$ or $-\infty$ as we did here, or treat those cases separately.

(b) For $a_n = -\frac{1}{n}$, $b_n = n$ for n even and $b_n = 1$ for n odd, we have $a_n b_n = -1$ for n even and $a_n b_n = -\frac{1}{n}$ for n odd, so
$$\liminf_{n\to\infty} b_n = 1, \quad \lim_{n\to\infty} a_n = 0, \quad \liminf_{n\to\infty} a_n b_n = -1$$
and
$$\liminf_{n\to\infty} a_n b_n = -1 \neq \lim_{n\to\infty} a_n \liminf_{n\to\infty} b_n = 0 \cdot 1 = 0.$$

3. Prove that if $f_n : E \to \mathbb{R}$ and (f_n) is uniformly convergent on every at most countable subset of E, then (f_n) is uniformly convergent on E.

Solution. First we need to find a function f that (f_n) converges to on E. Suppose (f_n) is uniformly convergent on every at most countable subset of E. In particular, (f_n) converges uniformly on any set $\{x\}$, which is finite, so the pointwise limit exists for all $x \in E$, and define function f by

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Since uniformly convergent sequence of functions implies pointwise convergence with the same limit, and limit is unique, f is the only possible uniform limit of (f_n) on E. Now, suppose that on the contrary (f_n) does not converge uniformly to f on E. Then,

$$\neg\left(\forall\varepsilon > 0 \exists N \forall n \geq N \forall x \in E : |f_n(x) - f(x)| < \varepsilon\right),$$

which is equivalent to

$$\exists\varepsilon > 0 \forall N \exists n \geq N \exists x \in E : |f_n(x) - f(x)| \geq \varepsilon.$$

So, taking such ε and $N = 1, 2, \ldots$ in turn, for each N, there exist n_N and x_N such that

$$x_N \in E, \quad n_N \geq N, \quad |f_{n_N}(x_N) - f(x_N)| \geq \varepsilon > 0,$$

which contradicts to uniform convergence of f_n to f on the set $\{x_N : N \in \mathbb{N}\}$.

4. Assume that $a_n \in \mathbb{R}$ for all n, and $\sum_{n=1}^{\infty} \frac{a_n}{n^x}$ converges for $x = x_0 \in \mathbb{R}$. Show that then the series converges for all $x > x_0$. (Be careful that there is **no assumption** on the signs of the a_n.)

Solution. Since there is no assumption on the signs of the a_n, this problem is about non-absolute convergence, and criteria such as the comparison test, which gives absolute convergence, will not be useful. The following threorem was used to prove the convergence of the alternating series: *If the partial sums of* $\sum_{n=1}^{\infty} b_n$ *are bounded and* $a_n \searrow 0$*, then* $\sum_{n=1}^{\infty} a_n b_n$ *converges.* So, for $x > x_0$, write

$$\sum_{n=1}^{\infty} \frac{a_n}{n^x} = \sum_{n=1}^{\infty} \frac{a_n}{n^{x_0}} \frac{1}{n^{x-x_0}}$$

Since the series $\sum_{n=1}^{\infty} \frac{a_n}{n^{x_0}}$ converges by assumption, it has bounded partial sums, and since $x > x_0$, $\frac{1}{n^{x-x_0}} \searrow 0$. Thus, $\sum_{n=1}^{\infty} \frac{a_n}{n^x}$ converges.

5. Let $\sum_{n\geq 0} u_n$ be a convergent series with real nonnegative terms, $u_n \geq 0$. For all $n \in \mathbb{N}$, we define $v_n = \sup_{p\geq n} u_p$. Does it follow that the series $\sum_{n\geq 0} v_n$ converge?

Solution. No. Counterexample:

$$\sum_{n\geq 0} u_n = 1 + \frac{1}{4} + 0 + \frac{1}{9} + 0 + 0 + \frac{1}{16} + 0 + 0 + 0 + \cdots = \sum_{n\geq 0} \frac{1}{(n+1)^2} < \infty$$

since the partial sums are nondecreasing, and the partial sums of $\sum_{n\geq 0} \frac{1}{(n+1)^2}$ are a subsequence of the partial sums of $\sum_{n\geq 0} u_n$. But

$$\sum_{n\geq 0} v_n = 1 + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\frac{1}{2}} + \underbrace{\frac{1}{9} + \frac{1}{9} + \frac{1}{9}}_{\frac{1}{3}} + \underbrace{\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}}_{\frac{1}{4}} + \cdots = \sum_{n\geq 0} \frac{1}{n+1} = \infty.$$

We have used that $\sum_{n=1}^{\infty} \frac{1}{n^x} < \infty$ iff $x > 1$.

6. Suppose that $A \subset [0,1]$ is a countable set with only a single limit point $x_0 \in (0,1)$. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ otherwise} \end{cases}$$

Using the definition of Riemann integral, find if the Riemann integral $\int_0^1 f(x)\,dx$ exists, and find its value if it does.

Solution. Let $\varepsilon > 0$. Construct a partition P as follows. Since $x_0 \in (0,1)$, there exist a_0, b_0 such that $0 \le a_0 < x_0 < b_0 \le 1$ and $b_0 - a_0 < \frac{\varepsilon}{2}$. There are only finitely many points of A outside of the interval $[a_0, b_0]$ since if there were infinitely many, they would have a limit point by Weierstrass theorem, which would be outside of the interval (a_0, b_0) and thus distinct from x_0. Denote $A \setminus [a_0, b_0] = \{x_1, \ldots, x_n\}$. Around each of the points x_k construct and interval $(a_k, b_k) \ni x_k$ such that $b_k - a_k < \frac{\varepsilon}{2n}$ and $b_k - a_k < \frac{|x_k - x_j|}{2}$ for all $k, j = 1, \ldots, n$ so that the intervals do not overlap. Define partition P by the points $a_0, b_0, \ldots, a_n, b_n$. Then, $L(P, f) = 0$ since each interval contains a point not in A, and

$$\begin{aligned} U(P,f) &= (b_0 - a_0) + (b_1 - a_1) + \cdots + (b_n - a_n) \\ &< \frac{\varepsilon}{2} + n\frac{\varepsilon}{2n} = \varepsilon. \end{aligned}$$

Thus, $f \in \mathcal{R}[0,1]$ and $\int_0^1 f(x)\,dx = 0$.

Note: The problem is asking to use the **definition** of Riemann integral. Thus, a solution invoking the theorem that bounded function whose set of discontinuities is countable is Riemann integrable is not sufficient.