1. Let (X, d) be a metric space.

   (a) Prove or find a counterexample: If (x_n) is a Cauchy sequence in (X, d), then (x_n) converges.

   (b) Prove that if (x_n) and (y_n) are both Cauchy sequences in (X, d), then the sequence $(d(x_n, y_n))$ converges.

2. Let (a_n), (b_n) be sequences in $\mathbb{R}$ and suppose $\lim_{n\to\infty} a_n = a \in \mathbb{R}$.

   (a) Prove that if $a > 0$, then

   $$\liminf_{n\to\infty} a_n b_n = a \liminf_{n\to\infty} b_n.$$

   Hint: There is no assumption that $\liminf_{n\to\infty} b_n$ *is finite. You can do the finite and infinite cases separately, or try to do them together.*

   Hint: You must clearly indicate where the assumption $a \neq 0$ *is used in the proof.*

   (b) Provide a counterexample when the statement fails with $a = 0$ and $\liminf_{n\to\infty} b_n \in \mathbb{R}$.

3. Prove that $f_n : E \to \mathbb{R}$ and (f_n) is uniformly convergent on every finite or countable subset of E, then (f_n) is uniformly convergent on E.

4. Assume that $a_n \in \mathbb{R}$ for all n, and $\sum\limits_{n=1}^{\infty} \frac{a_n}{n^x}$ converges for $x = x_0 \in \mathbb{R}$. Show that then the series converges for all $x > x_0$. (Be careful: there is **no assumption** on the signs of the a_n.)

5. Let $\sum_{n\geq 0} u_n$ be a convergent series with real nonnegative terms, $u_n \geq 0$. For all $n \in \mathbb{N}$, we define $v_n = \sup_{p\geq n} u_p$. Does it follow that the series $\sum_{n\geq 0} v_n$ converges?

6. Suppose that $A \subset [0, 1]$ is a countable set with only a single limit point $x_0 \in (0, 1)$. Define $f : [0, 1] \to \mathbb{R}$ by

   $$f(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

   Using the definition of Riemann integral, find if the Riemann integral $\int\limits_0^1 f(x)\, dx$ exists, and find its value if it does.