PHD PRELIMINARY EXAMINATION IN APPLIED ANALYSIS
JANUARY 25, 2019

Name: ____________________________________

- The examination consists of 6 problems.
- Each problem is worth 20 points. Unless specified otherwise, numbered parts of a problem have equal weight.
- Justify your solutions: cite theorems that you use, provide counter-examples, give explanations.
- Write legibly using a dark pencil or pen. Rewrite your solution if it gets too messy.
- Please begin solution to every problem on a new page; **write only on one side of paper**; number all pages throughout; and, just in case, write your name on every page.
- Do not submit scratch paper or multiple alternative solutions. If you do, we will grade the first solution to its end and we will not attempt to fish for the truth.
- Ask the proctor if you have any questions.

Good luck!

1. __________
2. __________
3. __________
4. __________
5. __________
6. __________

Total _______

Examination committee: Jan Mandel, Dmitriy Ostrovskiy, Burt Simon (chair).

(1) Let $\{x_n\}$ be a sequence of real numbers. Prove that $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$. Hint: You can use the fact that the infimum of a set is less than or equal to the supremum.

Solution. By definition,

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} a_n, \quad a_n = \inf\{x_n, x_{n+1}, \ldots\}$$
$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} b_n, \quad b_n = \sup\{x_n, x_{n+1}, \ldots\}$$

where $a_n, b_n \in [-\infty, \infty]$. We were not asked to prove the existence of $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$, so we just state this. As we are allowed to use, $a_n \leq b_n$, thus

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n = \limsup_{n\to\infty} x_n$$

by a standard property of limits (which holds also when one or both of the limits are infinite).

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and suppose $f'(x) > 0$, $x \in (a, b)$. Prove that f is strictly increasing on $[a, b]$.

Solution. Let $a \leq x < y \leq b$. We need to show that $f(x) < f(y)$. Since $f : \mathbb{R} \to \mathbb{R}$ is differentiable, f is continous on $\mathbb{R}$ and thus on $[x, y]$. The mean value theorem states that if $x < y$ and f is continous on $[x, y]$ and differentiable on (x, y), then there exists $\xi \in (x, y)$ such that $f'(\xi) = \frac{f(y)-f(x)}{y-x}$. Taking this ξ, we have $f'(\xi) > 0$ since $a \leq x < \xi < y \leq b$, and thus

$$f(y) - f(x) = \underbrace{f'(\xi)}_{>0} \underbrace{(y - x)}_{>0} > 0.$$

Note: we do not need to assume that $f(x) > 0$ at $x = a$ or $x = b$.

(3) Let $\{f_n\}$ be a sequence of real-valued functions on $D \subset \mathbb{R}$ such that $|f_n(x)| \leq M_n < \infty$ for all n and all $x \in D$.

(a) Prove that if $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on D. (This is the Weierstrass M-test.)

(b) Show that the converse is not true by constructing a counterexample.

Solution. For each $x \in D$, $a_N = \sum_{n=1}^{N} |f_n(x)|$ is a monotonic sequence bounded below by zero and above by $\sum_{n=1}^{\infty} M_n < \infty$, and therefore converges by the Monotonic Convergence Theorem (Rudin, Theorem 3.14). Thus, $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely, and therefore converges (Rudin, Theorem 3.45). Let $g(x) = \sum_{n=1}^{\infty} f_n(x)$. Choose $\epsilon > 0$. Since $\sum_{n=1}^{\infty} M_n$ converges, there exists N so that $\sum_{n=N+1}^{\infty} M_n < \epsilon$. Then for all $x \in D$,

$$\left| g(x) - \sum_{n=1}^{N} f_n(x) \right| = \left| \sum_{n=N+1}^{\infty} f_n(x) \right| \leq \sum_{n=N+1}^{\infty} |f_n(x)| \leq \sum_{n=N+1}^{\infty} M_n < \epsilon.$$

Hence, $\sum_{n=1}^{N} f_n \to g$ as $N \to \infty$, uniformly on D.

To show the converse is false, consider $f_n(x) = (-1)^{n+1}/n$. Then $M_n = 1/n$, so $\sum_{n=1}^{\infty} M_n$ diverges. But $\sum_{n=1}^{\infty} f_n$ converges uniformly since for all $x \in D$, $\sum_{n=1}^{N} f_n(x) = \sum_{n=1}^{N}(-1)^{n+1}/n$ converges as $N \to \infty$ by the alternating series theorem (Rudin, Theorem 3.43), since $c_n = (-1)^{n+1}/n$ is alternating sequence, $|c_n|$ is decreasing, and $c_n \to 0$ as $n \to \infty$.

(4) Let $\{x_n\}$ be a bounded sequence of real numbers.
   (a) Prove that $x_n \to x$ if and only if every convergent subsequence of $\{x_n\}$ converges to x.
   (b) Find a counterexample to part (a) if the sequence is not bounded.

Solution. Suppose $x_n \to x$ and $x_{n_i} \to y \neq x$. Let $\epsilon = |x - y|$. Since $x \neq y$, we have $\epsilon > 0$. There exists N_1 such tha that $n > N$ implies $|x_n - x| < \epsilon/2$ and N_2 such that $n_i > N_2$ implies $|x_{n_i} - y| < \epsilon/2$. Choose $N = \max\{N_1, N_2\}$. Then the reverse triangle inequality yields

$$|x_{n_i} - x| \geq |x - y| - |x_{n_i} - y| \geq \epsilon/2$$

which is a contradiction.

Conversely, suppose every convergent subsequence converges to x. If x_n does not converge to x then there exists $\epsilon > 0$ such that for all N, there exists an $n(N) > N$ with $|x_n - x| > \epsilon$. Construct a strictly increasing sequence of integer numbers according to the following recursive rule: $n_1 = 1$, $n_{i+1} = n(n_i)$. By construction, all elements of the subsequence $\{x_{n_i}\}$ satisfy $|x_{n_i} - x| > \epsilon$. Since $\{x_n\}$ is bounded, so is $\{x_{n_i}\}$. By the Bolzano-Weierstrass Theorem, there must be a convergent subsequence of $\{x_{n_i}\}$, but that convergent subsequence cannot converge to x since every element differs from x by at least ϵ. This contradicts the original assumption.

If $\{x_n\}$ is not bounded then the assertion in part (a) is false. For example, let $x_n = 0$ if n is odd and $x_n = n$ if n is even. Every convergent subsequence converges to 0, but the sequence itself does not converge.

(5) Let (X,d) be a metric space, and let $A \subset X$. Define ∂A (the boundary of A) to be the set of all points in X for which every neighborhood contains at least one point in A and at least one point in A^c. Prove that $\partial A = \bar{A} \cap \bar{A^c}$.

Solution. Let $x \in \partial A$. For all $x \in X$ either $x \in A$ or $x \in A^c$. Suppose $x \in A$ and therefore $x \in \bar{A}$. Because every neighborhood of x contains at least one point from A^c and $x \notin A^c$, x must be a limit point of A^c and thus $x \in \bar{A^c}$. Because $x \in \bar{A}$ and $x \in \bar{A^c}$, $x \in \bar{A} \cap \bar{A^c}$. If $x \in A^c$ it means that $x \in \bar{A^c}$. Because every neighborhood of x contains at least one point from A and $x \notin A$, x must be a limit point of A and thus $x \in \bar{A}$. Because $x \in \bar{A}$ and $x \in \bar{A^c}$, $x \in \bar{A} \cap \bar{A^c}$.
Let $x \in \bar{A} \cap \bar{A^c}$ and suppose $x \in A$. Then $x \in \bar{A^c} \setminus A^c$ (because $x \notin A^c$) and therefore is a limit point of A^c, which means that every neighborhood of x contains a point from A^c and also from A (x itself). If $x \in A^c$ then $x \in \bar{A} \setminus A$ and therefore is a limit point of A, which means that every neighborhood of x contains a point from A^c ($x \in A^c$) and also from A.

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be Riemann integrable on every interval $[0, t],\ t < \infty$ and define

$$I = \lim_{t \to \infty} \int_0^t f(x)dx$$

if the limit exists. We say that f is absolutely integrable on $[0, \infty)$ if

$$\lim_{t \to \infty} \int_0^t |f(x)|dx < \infty.$$

(a) Find an example of a continuous function f where I exists, but f is not absolutely integrable on $[0, \infty)$.
(b) Find an example of a continuous function f that is absolutely integrable on $[0, \infty)$, but is not bounded.
(c) Prove that if f is absolutely integrable on $[0, \infty)$ then I exists.

Solution.
(a) Define $f(x) = \frac{\sin(x)}{(x+1)}$. For $x \in [n\pi, (n+1)\pi]$,

$$\frac{|\sin(x)|}{(n+2)\pi} < |f(x)| < \frac{|\sin(x)|}{(n+1)\pi},$$

so

$$\frac{2}{(n+2)\pi} < \int_{n\pi}^{(n+1)\pi} |f(x)|dx < \frac{2}{(n+1)\pi}.$$

It follows that $f(x)$ is not absolutely integrable on $[0, \infty)$, since $\sum_{n=0}^{\infty} \frac{2}{(n+2)\pi} = \infty$. On the other hand, if $c_n = \int_0^{n\pi} f(x)dx$ then $\{c_n\}$ converges by the alternating series theorem. So, if $t \in (n\pi, (n+1)\pi)$ then $|c_n - \int_0^t f(x)dx| < \frac{2}{(n+1)\pi}$. It follows that $\int_0^t f(x)dx$ converges.

(b) Define $f(x)$ as follows

$$f(x) = \begin{cases} n(1 - n^3|x - n|), & x \in \left[n - \frac{1}{n^3}, n + \frac{1}{n^3}\right], n = 2, 3, ... \\ 0, & \text{otherwise} \end{cases}$$

$f(n) = n$ means that $f(x)$ is not bounded, but $\int_0^t |f(x)|dx = \int_0^t f(x)dx$ is monotonically increasing function bounded from above by $\sum_{n=2}^{\infty} \frac{1}{n^2}$ and therefore it has limit at $t \to \infty$. Note: $f(x)$ is zero everywhere except near integer points, n, where it looks like an isosceles triangle with height n and base $2/n^3$.

(c) Choose $\epsilon > 0$ and t^* large enough so that $\int_{t^*}^{\infty} |f(x)|dx < \epsilon$. Let $I_n = \int_0^n f(x)dx$. For any $t^* < n < m$, we have $|I_m - I_n| \leq \int_n^m |f(x)|dx < \epsilon$. Since ϵ was arbitrary, $\{I_n\}$ is a Cauchy sequence. Let $I = \lim I_n$. Since $\int_{\lfloor t \rfloor}^t f(x)dx \to 0$, $\int_0^t f(x)dx$ also converges to I.