PHD PRELIMINARY EXAMINATION IN APPLIED ANALYSIS
JANUARY 25, 2019

Name: ____

- The examination consists of 6 problems.
- Each problem is worth 20 points. Unless specified otherwise, numbered parts of a problem have equal weight.
- Justify your solutions: cite theorems that you use, provide counter-examples, give explanations.
- Write legibly using a dark pencil or pen. Rewrite your solution if it gets too messy.
- Please begin solution to every problem on a new page; **write only on one side of paper**; number all pages throughout; and, just in case, write your name on every page.
- Do not submit scratch paper or multiple alternative solutions. If you do, we will grade the first solution to its end and we will not attempt to fish for the truth.
- Ask the proctor if you have any questions.

Good luck!

1. __________
2. __________
3. __________
4. __________
5. __________
6. __________

Total _______

Examination committee: Jan Mandel, Dmitriy Ostrovskiy, Burt Simon (chair).

(1) Let $\{x_n\}$ be a sequence of real numbers. Prove that $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$. Hint: You can use the fact that the infimum of a set is less than or equal to the supremum.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and suppose $f'(x) > 0$, $x \in (a, b)$. Prove that f is strictly increasing on $[a, b]$.

(3) Let $\{f_n\}$ be a sequence of real-valued functions on $D \subset \mathbb{R}$ such that $|f_n(x)| \leq M_n < \infty$ for all n and all $x \in D$.

(a) Prove that if $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on D. (This is the Weierstrass M-test.)

(b) Show that the converse is not true by constructing a counterexample.

(4) Let $\{x_n\}$ be a bounded sequence of real numbers.

(a) Prove that $x_n \to x$ if and only if every convergent subsequence of $\{x_n\}$ converges to x.

(b) Find a counterexample to part (a) if the sequence is not bounded.

(5) Let (X, d) be a metric space, and let $A \subset X$. Define ∂A (the boundary of A) to be the set of all points in X for which every neighborhood contains at least one point in A and at least one point in A^c. Prove that $\partial A = \bar{A} \cap \bar{A}^c$.

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be Riemann integrable on every interval $[0, t],\ t < \infty$ and define

$$I = \lim_{t\to\infty} \int_0^t f(x)dx$$

if the limit exists. We say that f is absolutely integrable on $[0, \infty)$ if

$$\lim_{t\to\infty} \int_0^t |f(x)|dx < \infty.$$

(a) Find an example of a continuous function f where I exists, but f is not absolutely integrable on $[0, \infty)$.
(b) Find an example of a continuous function f that is absolutely integrable on $[0, \infty)$, but is not bounded.
(c) Prove that if f is absolutely integrable on $[0, \infty)$ then I exists.