Neumann and Dirichlet boundary conditions for Incompressible Navier-stokes problems

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Abstract

For my master's project, I analyzed the paper "On Boundary Conditions for Incompressible Navier-Stokes Problems", written by Dietmar Rempfer. Using the stream function that Kress and Montgomery suggested to Rempfer, I calculated the velocity field. We use the velocity field to derive the right-hand side of the Poisson equation for pressure, including alternative Neumann or Dirichlet boundary conditions. We confirmed the parameter values in the stream function that imply initial conditions that satisfy no-slip conditions at the boundary. Also, we checked the compatibility requirements for the Poisson equation source term with the Neumann boundary condition. Last, we derived the expressions for the pressure field solution from the Neumann and Dirichlet boundary conditions that contain the parameters of the stream function, and compare the answers with Rempfer after inputting the values of the parameters Rempfer picked.

1 Introduction

In Rempfer's paper, "On Boundary Conditions for Incompressible Navier-Stokes Problems" [**2**], Rempfer chose the stream function with the parameters that allow velocity field to satisfy no-slip boundary conditions at the boundary and used the stream function to find the pressure field. However, Rempfer only showed the expressions of the pressure field after inputting the values of the parameters of the stream function. The goal of this paper is to derive the pressure field from the Neumann and Dirichlet boundary conditions with the symbolic parameters of the stream function in both expressions. Section 2 provides some background knowledge on this article's topic. Section 3 illustrates the derivation of Neumann and Dirichlet boundary conditions. Section 4 shows the expressions of the pressure with Neumann and Dirichlet boundary conditions. Section 5 demonstrates the requirement of the compatibility condition for the Poisson equation with Neumann boundary condition. We compare the pressure fields expressions with Rempfer in section 6. Finally, we talk about the possible improvements for this project.

2 Background

The incompressible Navier-Stokes equation is given by

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 u$$
$$\nabla \cdot u = 0.$$

where $\nabla \cdot u = 0$ represents the incompressible flows, u is the velocity vector and p is the pressure divided by the constant density of the fluid. From Alexa Desautels and Dietmar Rempfer [**2**] [**6**], we get the Poisson equation by taking the divergence of the imcompressible Navier-Stokes equation

$$\begin{aligned} \nabla^2 p &= -\nabla \cdot (u \cdot \nabla u) \\ \nabla \cdot u &= 0. \end{aligned} \tag{1}$$

The stream function is defined for incompressible (e.g. divergence-free flows) in two dimensions or in three dimensions with axisymmetry. The stream function is related to the Chandrasekhar-Reid function and velocity fields can be obtained from it [**2**]. In Rempfer, they considered the case of two-dimensional flow in a channel with line boundary at $y = -1$ and $y = 1$, periodic in x with domain $0 < x \leq 4$.

$$\Psi(x, y) = \cos(\frac{\pi}{2}x)\Big(\cos(\lambda y) + A_\lambda \cosh(\frac{\pi}{2}y)\Big) \tag{2}$$

is the stream function he provided to construct the initial condition. λ and A_λ are the parameters that allow the velocity field to satisfy the no-slip boundary conditions at the boundary (e.g., u= 0 at the boundary). The term $\cosh(\frac{\pi}{2}y)$ adds a potential flow velocity component and the velocity has two properties. One is the velocity potential to satisfy the Poisson equation, the other is the velocity has two boundary conditions that both the normal and tangential velocity components disappear at the boundary.

Now in order to solve for p in (1), boundary conditions are necessary. In the following section, the Neumann and Dirichlet Boundary conditions are derived.

3 Derivation of the Neumann and Dirichlet Boundary Conditions

The initial time velocity field, $U_0 = \nabla \times (0, 0, \Psi)$ is defined by the stream function (2) above,

$$U_0(x,y) = \begin{pmatrix} u_0(x,y) \\ \\ v_0(x,y) \end{pmatrix} = \begin{pmatrix} \cos(\frac{\pi}{2}x)(\frac{\pi}{2}A_\lambda \sinh(\frac{\pi}{2}y) - \lambda \sin(\lambda y)) \\ \\ \frac{\pi}{2}\sin(\frac{\pi}{2}x)(\cos(\lambda y) + A_\lambda \cosh(\frac{\pi}{2}y)) \end{pmatrix}. \tag{3}$$

Substituting U_0 into (1) and letting $f(x, y) = -\nabla \cdot (U_0 \cdot \nabla U_0)$, we get

$$f(x,y) = -\Big(\frac{\partial}{\partial x}(u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y}) + \frac{\partial}{\partial y}(u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y})\Big)$$

$$-f(x,y) = \Big((\frac{\partial u_0}{\partial x})^2 + u_0 \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial v_0}{\partial x}\frac{\partial u_0}{\partial y} + v_0 \frac{\partial^2 u_0}{\partial x \partial y} \Big) + \Big((\frac{\partial v_0}{\partial y})^2 + v_0 \frac{\partial^2 v_0}{\partial y^2} + \frac{\partial v_0}{\partial x}\frac{\partial u_0}{\partial y} + u_0 \frac{\partial^2 v_0}{\partial x \partial y} \Big).$$

Substituting in $u_0(x,y)$ and $v_0(x,y)$, the right hand side of the Poisson equation [**9**]

$$\begin{aligned} f(x,y) = &\frac{\pi^2}{8} \Bigg(\cos^2(\frac{\pi}{2}x) \Big(\cos(\lambda y) + A_\lambda \cosh(\frac{\pi}{2}y) \Big) \Big(4\lambda^2 \cos(\lambda y) - \pi^2 A_\lambda \cosh(\frac{\pi}{2}y) \Big) \\ &- \sin^2(\frac{\pi}{2}x) \Big(2\lambda \sin(\lambda y) - \pi A_\lambda \sinh(\frac{\pi}{2}y) \Big)^2 \Bigg) \end{aligned} \tag{4}$$

and the problem becomes

$$\begin{aligned} \nabla^2 p = f(x,y) \qquad & \forall (x,y) \in \mathbb{R} \times (-1,1) \\ p(x,y) = p(x+4,y). & \end{aligned} \tag{5}$$

The incompressible flow differential equation is

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 u$$

$$\nabla \cdot u = 0.$$

A solid, stationary boundary (e.g., $\frac{\partial u}{\partial t} = 0$ at the boundary) with no-slip boundary conditions for the velocity. To check what drives the flow, consider the domain-total specific kinetic energy

$$E(t) = \int_0^4 \int_{-1}^1 \frac{u(x,y,t) \cdot u(x,y,t)}{2} dxdy.$$

Taking the derivative with expect to t on both side we have

$$\frac{dE(t)}{dt} = \int_0^4 \int_{-1}^1 \frac{\partial u(x,y,t)}{\partial t} \cdot u(x,y,t) dxdy = \int_0^4 \int_{-1}^1 (-\nabla p - u \cdot \nabla u + \frac{1}{\mathrm{Re}} \nabla^2 u) \cdot u \; dxdy.$$

Using the identity $(u \cdot \nabla) u = \frac{1}{2} \nabla (u \cdot u) - u \times (\nabla \times u)$, we have

$$\frac{dE(t)}{dt} = \int_0^4 \int_{-1}^1 (-\nabla p - \frac{1}{2} \nabla (u \cdot u) + u \times (\nabla \times u) + \frac{1}{\mathrm{Re}} \nabla^2 u) \cdot u \; dxdy.$$

Using the idetities $\nabla \cdot (pu) = (\nabla p) \cdot u + p(\nabla \cdot u)$, $\nabla \cdot u = 0$, $(u \times (\nabla \times u)) \cdot u = 0$, and $(\nabla^2 u) \cdot u = \nabla \cdot ((\nabla u) \cdot u) - ||\nabla u||_{F^2}$

$$\frac{dE(t)}{dt} = \int_0^4 \int_{-1}^1 -\nabla \cdot (pu) - \frac{1}{2} \nabla \cdot (u(u \cdot u)) + \frac{1}{\mathrm{Re}} (\nabla \cdot ((\nabla u) \cdot u) - ||\nabla u||_{F^2}) \; dxdy.$$

Apply Gauss Divergence theorem we have

$$\frac{dE(t)}{dt} = -\int_{\partial \Omega} (pu) \cdot \vec{\eta} d\sigma - \int_{\partial \Omega} \frac{1}{2} u (u \cdot u) \cdot \vec{\eta} d\sigma + \frac{1}{\mathrm{Re}} \int_{\partial \Omega} ((\nabla u) \cdot u) \cdot \vec{\eta} d\sigma - \frac{1}{\mathrm{Re}} \int_\Omega ||\nabla u||_{F^2} \; dA.$$

Since we have the no-slip boundary condition at the boundary which means $u = 0$ on the boundary $(\partial\Omega)$, then the first 3 integrals are equal to 0 (integral of 0 is 0 on the boundary $(\partial\Omega)$). And the last integral is negative because the norm is nonnegative inside the integral. We have $\frac{dE(t)}{dt} = -\frac{1}{\text{Re}}\int_\Omega ||\nabla u||_{F^2}\ dA < 0$ for $\text{Re} < \infty$. Therefore, the domain-total specific kinetic energy is decresing as time is increasing. The differential equation at the boundary is

$$\nabla p = \frac{1}{\text{Re}}\nabla^2 u. \tag{6}$$

We could get the Neumann boundary condition from (6) by projecting incompressible flow differential equation on the boundary normal, η and plugging in $u(x, y, t = 0) = U_0(x, y)$,

$$\frac{\partial p_{\text{Neu}}}{\partial \eta} = \eta \cdot \frac{1}{\text{Re}}\nabla^2 U_0. \tag{7}$$

We could get Dirichlet boundary condition by projecting (6) on the boundary-tangential coordinate τ and plugging in $u(x, y, t = 0) = U_0(x, y)$,

$$\frac{\partial p_{\text{Dir}}}{\partial \tau} = \tau \cdot \frac{1}{\text{Re}}\nabla^2 U_0. \tag{8}$$

where τ is a unit vector tangential to the boundary, and we could obtain the Dirichlet boundary condition after integrating the equation (8) along the boundary.

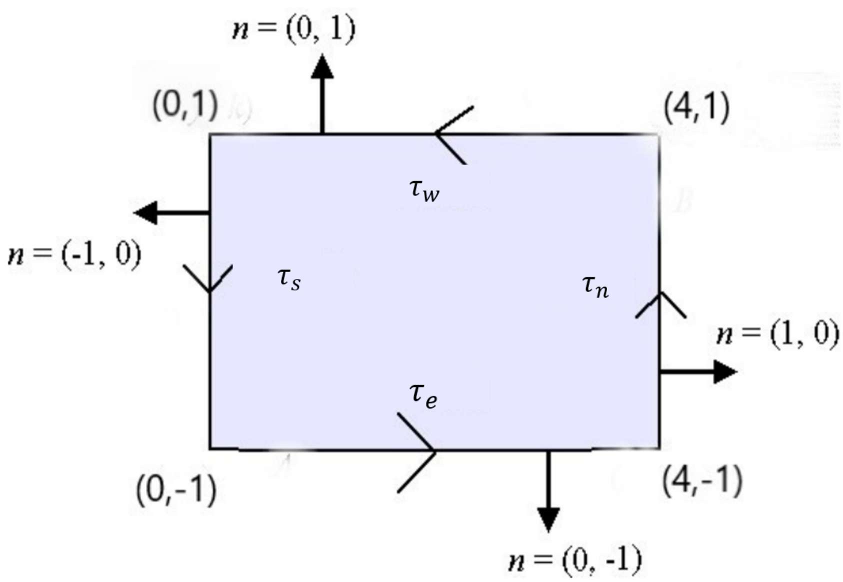

Figure 1: Boundary Condition

We will use Fig. 1 to calculate both of the boundary conditions. First Substituting U_0 in (6) [**10**] we obtain

$$\frac{1}{\text{Re}}\nabla^2 U_0 = \begin{pmatrix} \frac{\lambda(\pi^2+4\lambda^2)\cos(\frac{\pi}{2}x)\sin(\lambda y)}{4\text{Re}} \\ \\ \frac{-\pi(\pi^2+4\lambda^2)\sin(\frac{\pi}{2}x)\cos(\lambda y)}{8\text{Re}} \end{pmatrix}. \tag{9}$$

For the Neumann condition, we have

$$N_{\mathrm{n}} = \eta_{\mathrm{n}} \cdot \frac{1}{\mathrm{Re}} \nabla^2 U_0 = (0,1) \cdot \frac{1}{\mathrm{Re}} \nabla^2 U_0 = \frac{-\pi(\pi^2+4\lambda^2)\sin(\frac{\pi}{2}x)\cos(\lambda y)}{8\mathrm{Re}}$$

$$N_{\mathrm{s}} = \eta_{\mathrm{s}} \cdot \frac{1}{\mathrm{Re}} \nabla^2 U_0 = (0,-1) \cdot \frac{1}{\mathrm{Re}} \nabla^2 U_0 = \frac{\pi(\pi^2+4\lambda^2)\sin(\frac{\pi}{2}x)\cos(\lambda y)}{8\mathrm{Re}}.$$

Hence,

$$\frac{\partial p}{\partial y}(x,1) = N^+(x) = \frac{-\pi(\pi^2+4\lambda^2)\sin(\frac{\pi}{2}x)\cos(\lambda)}{8\mathrm{Re}}$$

$$\frac{\partial p}{\partial y}(x,-1) = N^-(x) = \frac{\pi(\pi^2+4\lambda^2)\sin(\frac{\pi}{2}x)\cos(\lambda)}{8\mathrm{Re}}.$$

For the Dirichlet condition [**11**], we have

$$\begin{aligned}
D_{\mathrm{e}} &= \int_0^x \tau_{\mathrm{e}} \cdot \frac{1}{\mathrm{Re}} \nabla^2 U_0(\xi,y) d\xi = \frac{-\lambda(\pi^2+4\lambda^2)\sin(\frac{\pi}{2}x)\sin(\lambda)}{2\pi\mathrm{Re}} && \text{for } y=-1 \\
D_{\mathrm{n}} &= D_{\mathrm{e}} + \int_{-1}^y \tau_{\mathrm{n}} \cdot \frac{1}{\mathrm{Re}} \nabla^2 U_0(x,\eta) d\eta = 0 && \text{for } x=4 \\
D_{\mathrm{w}} &= D_{\mathrm{n}} + \int_4^x \tau_{\mathrm{w}} \cdot \frac{1}{\mathrm{Re}} \nabla^2 U_0(\xi,y) d\xi = \frac{-\lambda(\pi^2+4\lambda^2)\sin(\frac{\pi}{2}x)\sin(\lambda)}{2\pi\mathrm{Re}} && \text{for } y=1 \\
D_{\mathrm{s}} &= D_{\mathrm{w}} + \int_1^y \tau_{\mathrm{s}} \cdot \frac{1}{\mathrm{Re}} \nabla^2 U_0(x,\eta) d\eta = 0 && \text{for } x=0.
\end{aligned}$$

Hence,

$$\begin{aligned}
p(x,1) = D^+(x) &= \frac{-\lambda(\pi^2+4\lambda^2)/\sin(\frac{\pi}{2}x)\sin(\lambda)}{2\pi\mathrm{Re}} \\
p(x,-1) = D^-(x) &= \frac{-\lambda(\pi^2+4\lambda^2)\sin(\frac{\pi}{2}x)\sin(\lambda)}{2\pi\mathrm{Re}}.
\end{aligned} \tag{10}$$

The Poisson equations have the domain $(x,y) \in \mathbb{R} \times (-1,1)$ with Neumann and Dirichlet boundary conditions becoming

$$\begin{aligned}
&\nabla^2 p_{\mathrm{Neu}} = f(x,y) \\
&p_{\mathrm{Neu}}(x,y) = p_{\mathrm{Neu}}(x+4,y) \\
&\frac{\partial p_{\mathrm{Neu}}}{\partial y}(x,\pm 1) = N^{\pm}(x)
\end{aligned} \tag{11}$$

$$\begin{aligned}
&\nabla^2 p_{\mathrm{Dir}} = f(x,y) \\
&p_{\mathrm{Dir}}(x,y) = p_{\mathrm{Dir}}(x+4,y) \\
&p_{\mathrm{Dir}}(x,\pm 1) = D^{\pm}(x).
\end{aligned} \tag{12}$$

In section 4, we will solve for p_{Neu} and p_{Dir}.

4 Pressure Determinations with Neumann and Dirichlet Boundary Conditions

Since p_{Neu} and p_{Dir} are both periodic in x, we use univariate Fourier series in x to solve both of the Poisson problems. We have

$$p(x,y) = \sum_{n\in\mathbb{Z}} p_n(y)e^{ix\lambda_n} \tag{13}$$

and since $0 < x \leq 4$,

$$\lambda_n = \frac{2n\pi}{4} = \frac{n\pi}{2},$$

and

$$p_n''(y) - \lambda_n^2 p_n(y) = f_n(y) \qquad n \in \mathbb{Z} \tag{14}$$

where $f_n(y) = \frac{1}{4}\int_0^4 f(x,y)e^{-ix\lambda_n}dx$ are the Fourier coefficients of $f(x,y)$.

Since the equations (14) are second order nonhomogeneous differential equations, then

$$p_n(y) = h_n(y) + q_n(y) \qquad n \in \mathbb{Z} \tag{15}$$

where $h_n(y)$ are homogeneous solutions and $q_n(y)$ are particular solutions. For homogeneous equation, we have

$$h_n''(y) - \lambda_n^2 h_n(y) = 0.$$

Therefore, the homogeneous solutions are

$$\begin{cases} h_n(y) = a_n \sinh(\lambda_n y) + b_n \cosh(\lambda_n y) & n \in \mathbb{Z}/\{0\} \\ h_0(y) = a_0 y + b_0 & n = 0. \end{cases}$$

For nonhomogeneous solutions, we have

$$q_n(y) = u_n(y)\cosh(\lambda_n y) + v_n(y)\sinh(\lambda_n y).$$

To get $u_n(y)$ and $v_n(y)$, we assume $u_n'(y)\cosh(\lambda_n y) + v_n'(y)\sinh(\lambda_n y) = 0$, (Variation of parameters [**8**]) and now we have

$$\begin{cases} u_n'(y)\cosh(\lambda_n y) + v_n'(y)\sinh(\lambda_n y) = 0 \\ \lambda_n(u_n'(y)\sinh(\lambda_n y) + v_n'(y)\cosh(\lambda_n y)) = q_n''(y) = f_n(y). \end{cases} \tag{16}$$

Hence, the solutions for (16) are

$$\begin{pmatrix} u_n(y) \\ v_n(y) \end{pmatrix} = \int_{-1}^{y} \frac{2f_n(w)}{n\pi} \begin{pmatrix} -\sinh(\lambda_n w) \\ \cosh(\lambda_n w) \end{pmatrix} dw.$$

Therefore, the (15) becomes

$$p_n(y) = \frac{2}{n\pi}\int_{-1}^{y} \Big(\sinh\big(\frac{n\pi}{2}(y-w)\big)\Big) f_n(w)dw + a_n \sinh(\frac{n\pi}{2}y) + b_n \cosh(\frac{n\pi}{2}y). \tag{17}$$

To get a_n and b_n, we need to use the boundary conditions. For Neumann boundary conditions, we have

$$\frac{\partial p}{\partial y}(x, \pm 1) = N^{\pm}(x).$$

Hence,

$$p_n'(\pm 1) = N_n^{\pm} \tag{18}$$

where $N_n^{\pm} = \frac{1}{4}\int_0^4 N^{\pm}(x) e^{\frac{-in\pi x}{2}} dx$ are the Fourier coefficients of $N^{\pm}(x)$. Taking the derivative of (17) and using (18), now we have

$$\begin{pmatrix} (a_n)_{\text{Neu}} \\ (b_n)_{\text{Neu}} \end{pmatrix} = \frac{\operatorname{csch}(\frac{n\pi}{2})\sinh(\frac{n\pi}{2})}{2n\pi} \begin{pmatrix} \sinh(\frac{n\pi}{2})\Big(\big(N_n^+ - \int_{-1}^1 \cosh(\frac{n\pi}{2}(1-w)) f_n(w) dw\big) + N_n^-\Big) \\ \cosh(\frac{n\pi}{2})\Big(\big(N_n^+ - \int_{-1}^1 \cosh(\frac{n\pi}{2}(1-w)) f_n(w) dw\big) - N_n^-\Big) \end{pmatrix}.$$

Therefore, using (17) with Neumann condition [**12**] we get,

$$\begin{aligned} \Big(p_n(y)\Big)_{\text{Neu}} &= \frac{1}{n\pi}\int_{-1}^{y}\Big(\sinh\big(\frac{n\pi}{2}(y-w)\big)\Big) f_n(w) dw + \frac{\operatorname{csch}(\frac{n\pi}{2})\sinh(\frac{n\pi}{2})}{2n\pi} \\ &\Big(\big(N_n^+ - \int_{-1}^{1}\cosh(\frac{n\pi}{2}(1-w)) f_n(w) dw\big)\cosh(\frac{n\pi}{2}(y+1)) - N_n^- \cosh(\frac{n\pi}{2}(y-1))\Big). \end{aligned} \tag{19}$$

For Dirichlet boundary condition, we have

$$p(x, \pm 1) = D^{\pm}(x).$$

Hence,

$$p_n(\pm 1) = D_n^{\pm} \tag{20}$$

where $D_n^{\pm} = \frac{1}{4}\int_0^4 D^{\pm}(x) e^{\frac{-in\pi x}{2}} dx$ are the Fourier coefficients of $N^{\pm}(x)$. We use (17) and (20) and get

$$\begin{pmatrix} (a_n)_{\text{Dir}} \\ (b_n)_{\text{Dir}} \end{pmatrix} = \frac{\operatorname{csch}(\frac{n\pi}{2})\sinh(\frac{n\pi}{2})}{2} \begin{pmatrix} \cosh(\frac{n\pi}{2})\Big(\big(D_n^+ - \frac{1}{n\pi}\int_{-1}^1 \sinh(\frac{n\pi}{2}(1-w)) f_n(w) dw\big) - D_n^-\Big) \\ \sinh(\frac{n\pi}{2})\Big(\big(D_n^+ - \frac{1}{n\pi}\int_{-1}^1 \sinh(\frac{n\pi}{2}(1-w)) f_n(w) dw\big) + D_n^-\Big) \end{pmatrix}.$$

Therefore, using (17) with Dirichlet condition [**12**] we get,

$$\begin{aligned} \Big(p_n(y)\Big)_{\text{Dir}} &= \frac{1}{n\pi}\int_{-1}^{y}\Big(\sinh\big(\frac{n\pi}{2}(y-w)\big)\Big) f_n(w) dw + \frac{\operatorname{csch}(\frac{n\pi}{2})\sinh(\frac{n\pi}{2})}{2} \\ &\Big(\big(D_n^+ - \frac{1}{n\pi}\int_{-1}^{1}\sinh(\frac{n\pi}{2}(1-w)) f_n(w) dw\big)\sinh(\frac{n\pi}{2}(y+1)) - D_n^- \sinh(\frac{n\pi}{2}(y-1))\Big). \end{aligned} \tag{21}$$

We use Fourier series method to find all the Fourier coefficients $f_n(y)$, $D_n^{\pm}$ and $N_n^{\pm}$. Let's use $D_n^{\pm}$ as an example, $f_n(y)$ and $N_n^{\pm}$. have the similar steps. From (10),

$$D^{\pm}(x) = \frac{-\lambda(\pi^2 + 4\lambda^2)\sin(\frac{\pi}{2}x)\sin(\lambda)}{2\pi\text{Re}}.$$

From Euler's formula, we have

$$\sin(\frac{\pi}{2}x) = \frac{e^{\frac{i\pi x}{2}} - e^{-\frac{i\pi x}{2}}}{2i}.$$

Then we have

$$\sum_{n\in\mathbb{Z}} D_n^{\pm} e^{\frac{in\pi x}{2}} = D^{\pm}(x) = \frac{-\lambda(\pi^2+4\lambda^2)\sin(\frac{\pi}{2}x)\sin(\lambda)}{2\pi\mathrm{Re}} = \frac{-\lambda(\pi^2+4\lambda^2)\sin(\lambda)}{2\pi\mathrm{Re}} \frac{e^{\frac{i\pi x}{2}} - e^{-\frac{i\pi x}{2}}}{2i}.$$

We could see that all Fourier coefficients except $D_1^{\pm}$ and $D_{-1}^{\pm}$ are equal to zero. Therefore the Fourier coefficients [**13**] for $D^{\pm}(x)$ are

$$D_n^{\pm} = \frac{i\lambda(\pi^2+4\lambda^2)\sin(\lambda)}{4\pi\mathrm{Re}} n\delta_{1,|n|}.$$

Similarly, the Fourier coefficients for $N^{\pm}(x)$ and $f(x,y)$ are

$$\begin{aligned} N_n^{\pm} =& \frac{\pm i\pi(\pi^2+4\lambda^2)\cos(\lambda)}{16\mathrm{Re}} n\delta_{1,|n|} \\ f_n(y) =& \frac{\pi^2}{16}\Bigg(\Big(\delta_{0,n} + \frac{1}{2}\delta_{2,|n|}\Big)\Big(\cos(\lambda y) + A_\lambda \cosh(\frac{\pi}{2}y)\Big)\Big(4\lambda^2\cos(\lambda y) - \pi^2 A_\lambda \cosh(\frac{\pi}{2}y)\Big) \\ & - \Big(\delta_{0,n} - \frac{1}{2}\delta_{2,|n|}\Big)\Big(2\lambda\sin(\lambda y) - \pi A_\lambda \sinh(\frac{\pi}{2}y)\Big)^2\Bigg). \end{aligned}$$

Tables 1 and 2 [**14**] are all the cases for $(p_n(y))_{\mathrm{Neu}}$ and $(p_n(y))_{\mathrm{Dir}}$.

Table 1: All cases of $(p_n(y))_{\mathrm{Neu}}$

n	$(p_n(y))_{\mathrm{Neu}}$
0	$\frac{\pi^2}{16}\Bigg(A_\lambda\Big(-A_\lambda\big(\pi(y+1)\sinh(\pi)+\cosh(\pi y)-\cosh(\pi)\big) -2\pi(y+1)\sinh(\frac{\pi}{2})\cos(\lambda)-4\cosh(\frac{\pi}{2}y)\cos(\lambda y)+4\cosh(\frac{\pi}{2})\big(\cos(\lambda)+\lambda(y+1)\sin(\lambda)\big)\Big) +\cos(2\lambda)+2\lambda(y+1)\sin(2\lambda)-\cos(2\lambda y)\Bigg)$
±1	$\frac{\pm i(4\lambda^2+\pi^2)\,\mathrm{csch}(\frac{\pi}{2})\cos(\lambda)\cosh(\frac{\pi}{2}y)}{8\mathrm{Re}}$
±2	$\frac{1}{32}\Bigg(\pi^2A_\lambda^2+\frac{4\pi A_\lambda}{4\lambda^2+9\pi^2}\Big(\pi(3\pi^2-4\lambda^2)\cosh(\frac{\pi}{2}y))\cos(\lambda y) -\frac{(4\lambda^2+\pi^2)\,\mathrm{csch}^2(\frac{\pi}{2})\cosh(\pi y)\big(3\pi\cos(\lambda)+2\lambda\coth(\frac{\pi}{2})\sin(\lambda)\big)}{\mathrm{csch}^2(\frac{\pi}{4})+\mathrm{sech}^2(\frac{\pi}{4})} +8\pi^2\lambda\sinh(\frac{\pi}{2}y)\sin(\lambda y)\Big)-4\lambda^2\Bigg)$
other	0

Table 2: All cases of $(p_n(y))_{\text{Dir}}$

n	$(p_n(y))_{\text{Dir}}$
0	$\frac{\pi^2}{16}\Big(A_\lambda\Big(A_\lambda\big(\cosh(\pi)-\cosh(\pi y)\big)+4\cosh(\frac{\pi}{2})\cos(\lambda)-4\cosh(\frac{\pi}{2}y)\cos(\lambda y)\Big) + \cos(2\lambda)-\cos(2\lambda y)\Big)$
±1	$\frac{\pm i\lambda(4\lambda^2+\pi^2)\operatorname{sech}(\frac{\pi}{2})\sin(\lambda)\cosh(\frac{\pi}{2}y)}{4\pi\text{Re}}$
±2	$\frac{\operatorname{sech}(\pi)}{32}\Big(\pi^2A_\lambda\Big(A_\lambda(\cosh(\pi)-\cosh(\pi y))+\frac{4}{4\lambda^2+9\pi^2}\big((3\pi^2-4\lambda^2)\cosh(\pi)\cosh(\frac{\pi}{2}y)\cos(\lambda y) + \cosh(\pi y)\big((4\lambda^2-3\pi^2)\cosh(\frac{\pi}{2})\cos(\lambda)-8\pi\lambda\sinh(\frac{\pi}{2})\sin(\lambda)\big) +8\pi\lambda\cosh(\pi)\sinh(\frac{\pi}{2}y)\sin(\lambda y)\big)\Big)+4\lambda^2\big(\cosh(\pi y)-\cosh(\pi)\big)\Big)$
other	0

Using Table 1,2 and (13) we have,

$$
\begin{aligned}
P_{\text{Dir}}(x,y) &= \frac{\pi^2}{16}\Big(A_\lambda\big(\cosh(\pi)A_\lambda+4\cosh(\frac{\pi}{2})\cos(\lambda)\big)+\cos(2\lambda)\Big)\\
&+\cos(\pi x)\Big(\frac{(4\lambda^2-\pi^2A_\lambda^2)\sinh^2(\pi y)}{16}-\frac{(\pi^2A_\lambda^2-4\lambda^2)\sinh(\pi)\sinh(\pi y)}{16}+\frac{(\pi^2A_\lambda^2-4\lambda^2)\cosh^2(\pi y)}{16}\\
&-\frac{(\pi^2A_\lambda^2-4\lambda^2)\cosh(\pi)\cosh(\pi y)}{16}+\frac{2\pi^3\lambda A_\lambda\sinh(\frac{\pi}{2}y)\sin(\lambda y)}{4\lambda^2+9\pi^2}\\
&+(\frac{3\pi^4}{4\lambda^2+9\pi^2}-\frac{\pi^2}{4})A_\lambda\cosh(\frac{\pi}{2}y)\cos(\lambda y)-\frac{\operatorname{sech}(\pi)}{32(4\lambda^2+9\pi^2)}\Big((4\lambda^2+9\pi^2)(\pi^2A_\lambda^2-4\lambda^2)\\
&\big(\cosh(\pi y)-\cosh(\pi(y+2))\big)+8\pi^2(3\pi^2-4\lambda^2)\cosh(\frac{\pi}{2})A_\lambda\cos(\lambda)\cosh(\pi y)\\
&+64\pi^3\lambda\sinh(\frac{\pi}{2})A_\lambda\sin(\lambda)\cosh(\pi y)\Big)\Big)-\frac{\pi^2A_\lambda^2\cosh(\pi y)}{16}-\frac{\pi^2A_\lambda\cosh(\frac{\pi}{2}y)\cos(\lambda y)}{4}\\
&-\frac{\lambda(4\lambda^2+\pi^2)\operatorname{sech}(\frac{\pi}{2})\sin(\lambda)\sin(\frac{\pi}{2}x)\cosh(\frac{\pi}{2}y)}{2\pi\text{Re}}+\frac{\pi^2(\sin^2(\lambda y)-\cos^2(\lambda y))}{16}
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
&P_{\text{Neu}}(x,y) = \\
&\cos(\pi x)\Bigg(\frac{(4\lambda^2-\pi^2A_\lambda^2)\sinh^2(\pi y)}{16} - \frac{(\pi^2A_\lambda^2-4\lambda^2)\sinh(\pi)\sinh(\pi y)}{16} + \frac{(\pi^2A_\lambda^2-4\lambda^2)\cosh^2(\pi y)}{16} \\
&- \frac{(\pi^2A_\lambda^2-4\lambda^2)\cosh(\pi)\cosh(\pi y)}{16} + \frac{2\pi^3\lambda A_\lambda \sinh(\frac{\pi}{2}y)\sin(\lambda y)}{4\lambda^2+9\pi^2} \\
&+ (\frac{3\pi^4}{4\lambda^2+9\pi^2} - \frac{\pi^2}{4})A_\lambda\cosh(\frac{\pi}{2}y)\cos(\lambda y) + \frac{1}{16(4\lambda^2+9\pi^2)} \\
&\Big(-\pi^2(4\lambda^2+9\pi^2)A_\lambda\cos(\lambda)\cosh(\pi(y+\frac{1}{2})) \\
&+ \pi A_\lambda\Big(2\lambda\sin(\lambda)\big((4\lambda^2+9\pi^2)\sinh(\pi(y+\frac{1}{2})) + (4\lambda^2-7\pi^2)\sinh(\pi(y+\frac{3}{2}))\big) \\
&+ \pi(20\lambda^2-3\pi^2)\cos(\lambda)\cosh(\pi(y+\frac{3}{2}))\Big) + \cosh(\pi(y+1))\Big((4\lambda^2+9\pi^2)(\pi^2A_\lambda^2-4\lambda^2) \\
&+ \pi A_\lambda\big(\pi\,\text{sech}(\frac{\pi}{2})\cos(\lambda)\big(3\pi^2(2+\cosh(\pi)) - 4\lambda^2(2+5\cosh(\pi))\big) \\
&- 2\lambda\,\text{csch}(\frac{\pi}{2})\sin(\lambda)\big(4\lambda^2\cosh(\pi)+\pi^2(8-7\cosh(\pi))\big)\big)\Big)\Big) \\
&- \frac{\pi^2A_\lambda^2\cosh(\pi y)}{16} - \frac{\pi^2A_\lambda\cosh(\frac{\pi}{2}y)\cos(\lambda y)}{4} + \frac{\pi^2}{16}\Big(\cosh(\pi)A_\lambda^2 \\
&+ 2A_\lambda\cos(\lambda)(2\cosh(\frac{\pi}{2}) - \pi(y+1)\sinh(\frac{\pi}{2})) \\
&+ (y+1)\big(4\lambda\sin(\lambda)(\cosh(\frac{\pi}{2})A_\lambda+\cos(\lambda)\big) - \pi\sinh(\pi)A_\lambda^2\big) + \cos(2\lambda)\Big) \\
&- \frac{(4\lambda^2+\pi^2)\,\text{csch}(\frac{\pi}{2})\cos(\lambda)\sin(\frac{\pi}{2}x)\cosh(\frac{\pi}{2}y)}{4\text{Re}} - \frac{\pi^2\cos(2\lambda y)}{16}.
\end{aligned}
\tag{23}
$$

In order to have an initial condition that is both divergence-free and that satisfies no-slip conditions at the boundary, we need to find the relationship between λ and A_λ. In the next section, we will show how to find λ, A_λ and compatibility requirements for p_{Neu}.

5 Compatibility

In order to have no-slip condition at the boundary, we need to have velocity field equal to zero at the boundary.($y = \pm 1$), so we have (3) equal to zero at $y = \pm 1$:

$$
\begin{pmatrix}
\cos(\frac{\pi}{2}x)(\frac{\pi}{2}A_\lambda\sinh(\frac{\pi}{2}) - \lambda\sin(\lambda)) \\
\\
\frac{\pi}{2}\sin(\frac{\pi}{2}x)(\cos(\lambda) + A_\lambda\cosh(\frac{\pi}{2}))
\end{pmatrix} = 0 \qquad \forall x \in (0,4].
$$

Then we have

$$
\begin{cases}
A_\lambda = -\cos(\lambda)\,\text{sech}(\frac{\pi}{2}) \\
2\lambda\sin(\lambda) = A_\lambda\pi\sinh(\frac{\pi}{2}).
\end{cases}
$$

There are infinitely many solutions for A_λ and λ. Rempfer chose $A_\lambda = 0.349911$ and $\lambda = 2.64244$ the smallest $|\lambda|$ case. They give the initial condition that is both divergence-free and that satisfies our no-slip conditions at the boundary.

Now we will find the arbitrary constant, first we note that

$$f_0(y) = p_0''(y).$$

Next integrating both sides from -1 to y, we get

$$\int_{-1}^{y} f_0(w)dw = p_0'(y) - p_0'(-1) = p_0'(y) - N_0^- = p_0'(y). \tag{24}$$

Integrating both sides from -1 to y again, we get

$$\int_{-1}^{y}\int_{-1}^{z} f_0(w)dwdz = \int_{-1}^{y}(y-w)f_0(w)dw = p_0(y) - p_0(-1).$$

We could see that $p_0(-1)$ is an arbitrary constant. In our case, we pick $p_0(-1) = 0$, so we have

$$p_0(y) = \int_{-1}^{y}(y-w)f_0(w)dw.$$

Plugging $y = 1$ in (24) we get

$$N_0^+ = p_0'(1) = \int_{-1}^{1} f_0(w)dw + N_0^-$$

$$N_0^+ - N_0^- = \int_{-1}^{1} f_0(w)dw.$$

Next we will find the requirement for the compatibility condition. The Poisson problem with Neumann boundary conditions has a solution if and only if the following compatibility condition holds [**4**] [**5**],

$$\int_{\Omega} f(x,y)dA = \int_{\partial\Omega} \frac{\partial}{\partial\vec{\eta}}p(x,y)d\sigma.$$

We could show the equality by using Gauss-Green (divergence) theorem [**7**],

$$\int_{\Omega} f(x,y)dA = \int_{\Omega} \nabla\cdot\nabla p(x,y)dA \overset{\text{Gauss}}{=} \int_{\partial\Omega} \nabla p(x,y)\cdot\vec{\eta}d\sigma = \int_{\partial\Omega} \frac{\partial}{\partial\vec{\eta}}p(x,y)d\sigma.$$

From Fig. 1, we get

$$\begin{aligned}\int_{\partial\Omega} \nabla p(x,y)\cdot\vec{\eta}d\sigma_A = &\int_{-1}^{1}\nabla p(4,y)\cdot(1,0)dy \text{ (at } x=4) + \int_{-1}^{1}\nabla p(0,y)\cdot(-1,0)dy \text{ (at } x=0)\\ &+\int_{0}^{4}\nabla p(x,1)\cdot(0,1)dx \text{ (at } y=1) + \int_{0}^{4}\nabla p(x,-1)\cdot(0,-1)dx \text{ (at } y=-1).\end{aligned}$$

Since $p(0,y) = p(4,y)$, we have

$$\int_{-1}^{1}\nabla p(4,y)\cdot(1,0)dy + \int_{-1}^{1}\nabla p(0,y)\cdot(-1,0)dy = 0.$$

Hence

$$\begin{aligned}\int_{\partial\Omega} \nabla p(x,y)\cdot\vec{\eta}d\sigma_A &= \int_{0}^{4}\nabla p(x,1)\cdot(0,1)dx + \int_{0}^{4}\nabla p(x,-1)\cdot(0,-1)dx\\ &= \int_{0}^{4}\frac{\partial}{\partial\vec{\eta}}p(x,1)dx - \int_{0}^{4}\frac{\partial}{\partial\vec{\eta}}p(x,-1)dx.\end{aligned}$$

Therefore,

$$\int_0^4 \int_{-1}^1 f(x,y)dydx = \int_0^4 N^+(x) - N^-(x)dx$$

$$\int_{-1}^1 f(x,y)dy = N^+(x) - N^-(x).$$

So we have the requirement

$$\int_{-1}^1 f_0(w)dw = N_0^+ - N_0^- = 0.$$

6 Results

After plugging in $A_\lambda = 0.349911$ and $\lambda = 2.64244$, the pressure field with Neumann boundary conditions

$$\begin{aligned} P_{\text{Neu}} &= -0.6168503cos(2\lambda y) - 1.670097cos(\pi x) - 0.07552573cosh(\pi y) \\ &- 0.8633708cos(\lambda y)cosh(\frac{\pi}{2}y) + 0.04890236cos(\pi x)cosh(\pi y) + 3.60531cosh(\frac{\pi}{2}y)\sin(\frac{\pi}{2}x) \\ &+ 0.01241453cos(\lambda y)cos(\pi x)cosh(\frac{\pi}{2}y) + 0.4910906cos(\pi x)\sin(\lambda y)\sinh(\frac{\pi}{2}y) - 0.692376 \end{aligned}$$

$$\begin{aligned} (P_{\text{Neu}})_{\text{Rempfer}} &= 0.0489026\cos(\pi x)\cosh(\pi y) - 1.6701\cos(\pi x) - 0.0755258\cosh(\pi y) \\ &- 0.863371\cos(2\lambda y)\cosh(\frac{\pi}{2}y) - 0.61685\cos(2\lambda y) + 3.30663\sin(\frac{\pi}{2}x)\sinh(\frac{\pi}{2}y) \\ &+ 0.0124142\cos(\lambda y)\cos(\pi x)\cosh(\frac{\pi}{2}y) + 0.491091\cos(\pi x)\sin(\lambda y)\sinh(\frac{\pi}{2}y) - 0.692376. \end{aligned}$$

Pressure field with Dirichlet boundary conditions

$$\begin{aligned} P_{\text{Dir}} &= 0.09976452\cos(\pi x)\cosh(\pi y) - 1.670097\cos(\pi x) - 0.07552573\cosh(\pi y) \\ &- 0.8633708\cos(\lambda y)\cosh(\frac{\pi}{2}y) - 0.6168503\cos(2\lambda y) + 3.032685\cosh(\frac{\pi}{2}y)\sin(\frac{\pi}{2}x) \\ &+ 0.01241453\cos(\lambda y)\cos(\pi x)\cosh(\frac{\pi}{2}y) + 0.4910906\cos(\pi x)\sin(\lambda y)\sinh(\frac{\pi}{2}y) - 0.692376 \end{aligned}$$

$$\begin{aligned} (P_{\text{Dir}})_{\text{Rempfer}} &= 0.0997648\cos(\pi x)\cosh(\pi y) - 1.6701\cos(\pi x) - 0.0755258\cosh(\pi y) \\ &- 0.863371\cos(\lambda y)\cosh(\frac{\pi}{2}y) - 0.61685\cos(2\lambda y) + 3.30663\sin(\frac{\pi}{2}x)\sinh(\frac{\pi}{2}y) \\ &+ 0.0124142\cos(\lambda y)\cos(\pi x)\cosh(\frac{\pi}{2}y) + 0.491091\cos(\pi x)\sin(\lambda y)\sinh(\frac{\pi}{2}y) - 0.692376. \end{aligned}$$

We could see that all coefficients are similar. The red term in $(P_{\text{Neu}})_{\text{Rempfer}}$ is a typo mistake that Rempfer made in his paper. The difference between our pressure field and Remfer's is the blue term. So

$$P_{\text{Rempfer}} = P + \delta_p$$

where $\delta_p = a\sin(\frac{\pi}{2}x)(\sinh(\frac{\pi}{2}y) - \cosh(\frac{\pi}{2}y))$ and a is the constant coefficient. And we have

$$\nabla^2\delta_p = 0.$$

Therefore,

$$\nabla^2 P_{\text{Rempfer}} = \nabla^2 P = f(x,y).$$

We know that P_{Dir} is an even function for y and $\frac{\partial P_{\text{Neu}}}{\partial y}$ is an odd function for y. Since δ_p and $\frac{\partial \delta_p}{\partial y}$ are neither, so Prempfer's pressure field expressions do not satisfy both of the boundary condtions.

7 Possible Improvements

Brian T. Kress and David C. Montgomery produced a stream function to obtain a 2-D solenoidal velocity field with the normal and tangential components that disappear at the boundary [**3**]. The stream function is

$$\Psi(x,y) = C_{k\lambda}\cos(kx)\Big(\cos(\lambda y) + A_\lambda \cosh(ky)\Big).$$

In our case, $C_{k\lambda} = 1$ and $k = \frac{1}{2}$. I could produce different stream functions by switching these two variables and get better expressions for both pressure fields from the Neumann and Dirichlet boundary conditions.

I got all of my results from MATLAB. The expression for both pressure fields are lengthy and it is difficult to input values of parameters of steam function and compare the results with Rempfer's pressure field expressions due to MATLAB's restrictions. Using different methods or program language to obtain shorter expressions of pressure field will help us compare the results in an easier and more meaningful manner.

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