Ambiguities in Pressure Determinations for Incompressible Flows

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Abstract

For my master's project, I analyzed the paper "Pressure determinations for incompressible fluids and magnetofluids" by Brian T. Kress and David C. Montgomery. Taking the divergence of the equation of motion for Newtonian fluids with a constant density yields a Poisson equation. When we assume the fluid to obey no-slip boundary conditions, ambiguities arise when solving the Poisson equation for pressure P. The no-slip boundary condition allows us to use the equation of motion to assume both Dirichlet and Neumann boundary conditions, which results in over-determination. When this problem has occurred, it has often been solved well enough for the application; however, when considering simple solenoidal velocity fields that vanish at the walls and that imply explicit solutions for the pressure, solving for P gives two different solutions that are non-trivially different at the initial moment, leading to different pressure gradients by the walls. This paper suggests replacing the no-slip boundary condition, where the disappearance of the tangential velocity at the walls is replaced with a wall friction term.

1 Introduction

An open question exists in the field of hydrodynamics concerning pressure determination. Taking the divergence of the equation of motion for a uniform density, Newtonian fluid results in a Poisson equation for the pressure. In order to solve this equation, we must use boundary conditions. In their paper "Pressure determinations for incompressible fluids and magnetofluids" [3], Brian T Kress and David C Montgomery analyze a stream function that produces a 2-D, solenoidal velocity field, the normal and tangential components of which disappear at the wall. This allows the pressure to obey both Neumann and Dirichlet boundary conditions. However, the pressure determined by each condition varies non-trivially near the wall at the initial instant.

The goal of this paper is to discuss this problem at length. Section 2 of this paper provides a background for the topics discussed. Section 3 presents a stream function that is used to determine pressures using both Neumann and Dirichlet conditions and shows the differences at $t = 0$. Finally, Section 4 contains a possible modification to the no-slip condition where a wall friction term is added.

2 Background

2.1 Chandrasekhar-Reid Functions

The stream function that the velocity fields are obtained from is related to the Chandrasekhar-Reid functions [1]. These functions are notable in that the normal component of the velocity and its derivative, related to the tangential component, disappear at the walls, allowing the functions to obey four boundary conditions. Chandrasekhar presents a proof that they form an orthogonal set and believes the set is complete, the proof of which is given by Erling Dahlberg[2]. The following characteristic value problem

$$\frac{d^4y}{dx^4} = \alpha^4 y$$

with 4 boundary conditions $y = 0$, $\frac{dy}{dx} = 0$ at $x = \pm\frac{1}{2}$ is considered. The standard forms of the solutions that form a set of orthogonal functions whose normal and tangential components disappear at the walls are

$$C_m(x) = \frac{\cosh(\lambda_m x)}{\cosh(\frac{1}{2}\lambda_m)} - \frac{\cos(\lambda_m x)}{\cos(\frac{1}{2}\lambda_m)},$$
$$S_m(x) = \frac{\sinh(\mu_m x)}{\sinh(\frac{1}{2}\mu_m)} - \frac{\sin(\mu_m x)}{\sin(\frac{1}{2}\mu_m)},$$

where $\alpha \to \lambda_m$ and $\alpha \to \mu_m$ are the roots of the characteristic equations

$$\tanh\frac{1}{2}\lambda + \tan\frac{1}{2}\lambda = 0,$$
$$\tanh\frac{1}{2}\mu - \tan\frac{1}{2}\mu = 0.$$

2.2 Derivation of the Poisson Equation

The continuity equation for fluid dynamics states that the rate at which mass enters a system is equal to the rate mass leaves the system plus the accumulation of mass in the system. Assuming no external source or sink of mass, this can be written as

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0,$$

where ρ is the density of the fluid and $\mathbf{v}$ is the velocity. For incompressible fluids, the density carried by each fluid parcel is constant. This simplifies the equation to $\nabla\cdot\mathbf{v} = 0$.

In deriving the Poisson equation, we start with the equation of motion, $\mathbf{M}$, for the fluid,

$$\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \frac{\mathbf{j}\times\mathbf{B}}{\rho c} - \nabla P + \nu\nabla^2\mathbf{v},$$

where $\mathbf{v} = (v_x, v_y)$ is the velocity vector, P is the pressure, ν is the kinematic viscosity, $\mathbf{B}$ is the magnetic field, $\mathbf{j} = \frac{c\nabla\times\mathbf{B}}{4\pi}$ is the electric current density, and ρ is the mass density. The paper focuses on the Navier-Stokes case for simplicity, so the magnetic terms can be dropped. The equation of motion can be written in vector form for its x and y components:

$$M_x = \frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} = -\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right),$$
$$M_y = \frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} = -\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right).$$

To derive the Poisson equation, we must take the divergence of the equation of motion, $\nabla\cdot\mathbf{M} = \frac{\partial}{\partial x}M_x + \frac{\partial}{\partial y}M_y$.

$$\frac{\partial}{\partial x}M_x = \frac{\partial}{\partial x}\frac{\partial v_x}{\partial t} + \frac{\partial v_x}{\partial x}\frac{\partial v_x}{\partial x} + v_y\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial v_y}{\partial x}\frac{\partial v_x}{\partial y} + v_y\frac{\partial^2 v_x}{\partial x\partial y} = -\frac{\partial^2 P}{\partial x^2} + \nu\left(\frac{\partial^3 v_x}{\partial x^3} + \frac{\partial^3 v_x}{\partial y^2\partial x}\right),$$
$$\frac{\partial}{\partial y}M_y = \frac{\partial}{\partial y}\frac{\partial v_y}{\partial t} + \frac{\partial v_x}{\partial y}\frac{\partial v_y}{\partial x} + v_x\frac{\partial^2 v_y}{\partial x\partial y} + \frac{\partial v_y}{\partial y}\frac{\partial v_y}{\partial y} + v_y\frac{\partial^2 v_y}{\partial y} = -\frac{\partial^2 P}{\partial x^2} + \nu\left(\frac{\partial^3 v_y}{\partial y^3} + \frac{\partial^3 v_y}{\partial x^2\partial y}\right).$$

We now add the equations. After some algebraic manipulation, the right hand side is

$$\frac{\partial}{\partial x}M_x + \frac{\partial}{\partial y}M_y = \frac{\partial}{\partial t}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) + \left(\frac{\partial v_x}{\partial x}\right)^2 + 2\frac{\partial v_x}{\partial y}\frac{\partial v_y}{\partial x} + v_x\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) + \left(\frac{\partial v_y}{\partial y}\right)^2 + v_y\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right),$$

while the left hand side is

$$\frac{\partial}{\partial x}M_x + \frac{\partial}{\partial y}M_y = -\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right) + \nu\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)\right].$$

Now rewrite each side and equate them:

$$\frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{v}\right) + \nabla\cdot\left(\mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla^2 P + \nu\nabla^2\left(\nabla\cdot\mathbf{v}\right).$$

We can now use the fact that for incompressible flows, $\nabla\cdot\mathbf{v} = 0$, and we are left with $\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla^2 P$ or

$$\nabla^2 P = -\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{v}).$$

Now we must solve the equation for P. To do this, we must use boundary conditions.

3 Pressure Determinations with No-Slip Boundary Conditions

We will now consider the case where the fluid obeys the no-slip condition i.e., the fluid has zero velocity relative to the boundary around the boundary. In this case, the left hand side of the equation of motion becomes zero, leaving the equation

$$\nabla P = \nu\nabla^2\mathbf{v}$$

Knowing the normal component of ∇P is sufficient to solve for P using Neumann boundary conditions, but it is also possible to solve for P using Dirichlet boundary conditions with knowledge of the tangential component of ∇P, using expressions like

$$P(x,y) - P(a,b) = \int_{\text{boundary from } (a,b) \text{ to } (x,y)} \nabla' P\cdot(dx', dy').$$

Now a stream function is considered that produces 2-D solenoidal velocity fields,

$$\psi(x,y) = C_{k\lambda}\cos(kx)[\cos(\lambda y) + A_{k\lambda}\cosh(ky)].$$

The term $A_{k\lambda}\cosh(ky)$ adds a potential flow velocity component to the stream function. There are two things to note about this. The first is that potential flow forces the velocity potential to satisfy Laplace's equation, the solutions of which will be used later. The second is that it allows us to force $\mathbf{v}$ to obey two boundary conditions as both the normal and tangential velocity components disappear at the boundary.

The 2-D velocity field, $\mathbf{v} = \nabla\psi\times\hat{\mathbf{e}}_z$, is in the xy-plane and has a wavenumber k in the x-direction. It is periodic in x. The values for λ and $A_{k\lambda}$ can be numerically solved in order to ensure that all spatial derivatives exist and the normal and tangential velocity components disappear when placed at the walls at $y = \pm a$. Additionally, we can find values for λ and $A_{k\lambda}$ that allow the term $\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{v})$ to be written as products of e^{ikx} and $e^{\lambda x}$ and their inverses. Finally, $C_{k\lambda}$ is a normalizing constant. Now we must find an inhomogeneous solution for P and add it to a solution of Laplace's equation. The inhomogeneous solution for P is the same for all boundary conditions while the solution for P for Laplace's equation can only satisfy either the normal component or the tangential component and must cancel the inhomogeneous solution at the boundary.

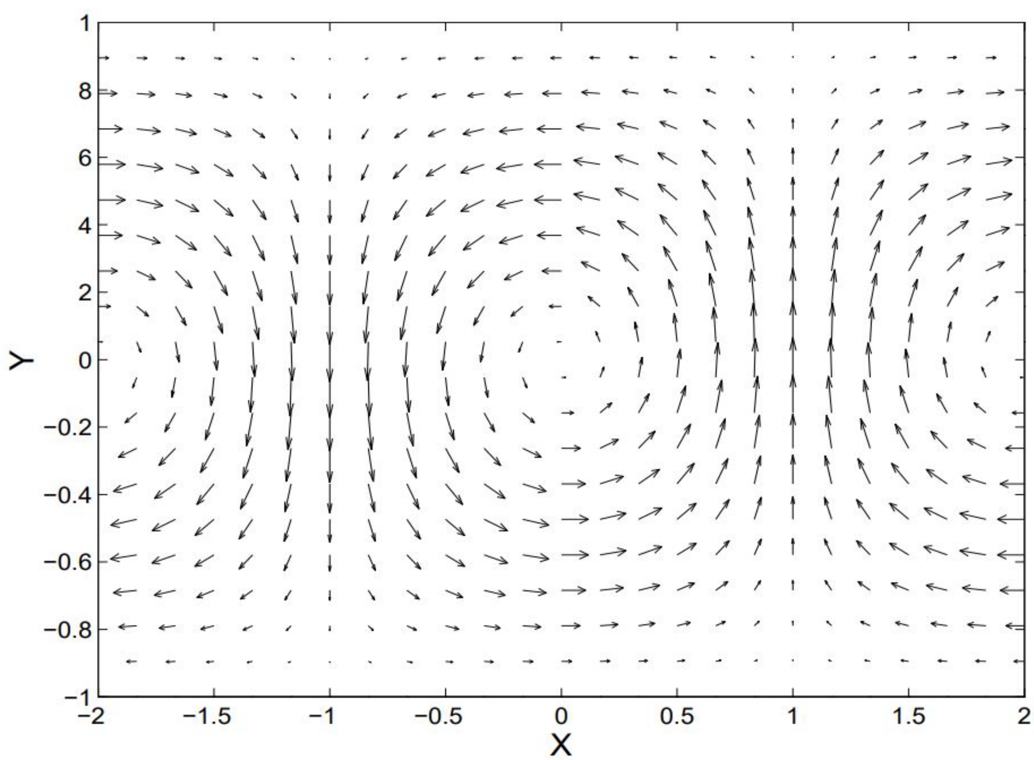

Figure 1: 2-D velocity field for $k = \frac{\pi}{2}$, $\lambda = 2.6424$, $A_{k\lambda} = 0.3499$ from $y = -1$ to $y = 1$

Now a specific function is chosen with values $k = \frac{\pi}{2}$, $\lambda = 2.6424$, $A_{k\lambda} = 0.3499$ with the wall at $y = \pm 1$. Figure 1 is the chosen velocity field.

We now want to solve the Poisson equation

$$\nabla^2 P = -\nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) \equiv b(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \Omega = [-2, 2] \times [-1, 1],$$

using Neumann boundary conditions

$$\hat{\mathbf{n}} \cdot \nabla P = \hat{\mathbf{n}} \cdot \nu \nabla^2 \mathbf{v} \quad \text{on} \quad \partial\Omega = \{[x, -2]\} \cup \cdots ,$$

or Dirichlet boundary conditions that can be determined by tangential gradients and formulas like

$$P(x, -1) - P(-2, -1) = \nu \int_{-2}^{x} \nabla'^2 \mathbf{v}(\mathbf{x}') \cdot (dx', 0).$$

The problem can be split up into two subproblems. The first subproblem is the non-homogeneous Poisson equation

$$\nabla^2 P = b,$$

that can be solved using homogeneous boundary conditions, arbitrarily Dirichlet,

$$P(x, \pm 1) = 0,$$
$$P(\pm 2, y) = 0,$$

to be canceled in subproblem 2. The choice of Neumann or Dirichlet boundary conditions does not play a role in this term of the solution.

The second subproblem is the homogeneous Laplace equation. Here a choice must be made to solve the problem using Neumann or Dirichlet boundary conditions, including the original prescribed conditions and conditions canceling boundary values from subproblem one. The homogeneous Laplace equation is

$$\nabla^2 P = 0,$$

with non-homogeneous boundary conditions e.g., Dirichlet:

$$P(x, -1) = f_1, \quad P(x, 1) = f_2,$$
$$P(-2, y) = f_3, \quad P(2, y) = f_4.$$

Subproblem two can be solved by separation of variables. The homogeneous Laplace equation must be split into four sub-subproblems, each having only one non-homogeneous boundary condition. For example, f_1 leads to a general solution [6]

$$P_1(x, y) = -\sum_{n=0}^{\infty} \left[\operatorname{csch} 2\pi(n+1) \int_0^2 f_1(x') \sin \frac{(n+1)\pi x'}{2} dx' \right]$$
$$\sinh \frac{\pi(n+1)(x-4)}{2} \sin \frac{(n+1)\pi y}{2}.$$

The four solutions can be combined to form $P(x, y) = P_1(x, y) + P_2(x, y) + P_3(x, y) + P_4(x, y)$. Now solve the Poisson equation with homogeneous Dirichlet boundary conditions. The general solution is [7]

$$P(x, y) = \sum_{m,n=0}^{\infty} c_{mn} \sin \frac{(m+1)\pi x}{4} \sin \frac{(n+1)\pi y}{2}$$

where

$$c_{mn} = -\frac{1}{\frac{(m+1)^2\pi^2}{8} + \frac{(n+1)^2\pi^2}{2}} \cdot$$
$$\int_0^2 \int_0^4 b(\mathbf{x}) \sin \frac{(m+1)\pi x}{4} \sin \frac{(n+1)\pi y}{2} dx dy$$

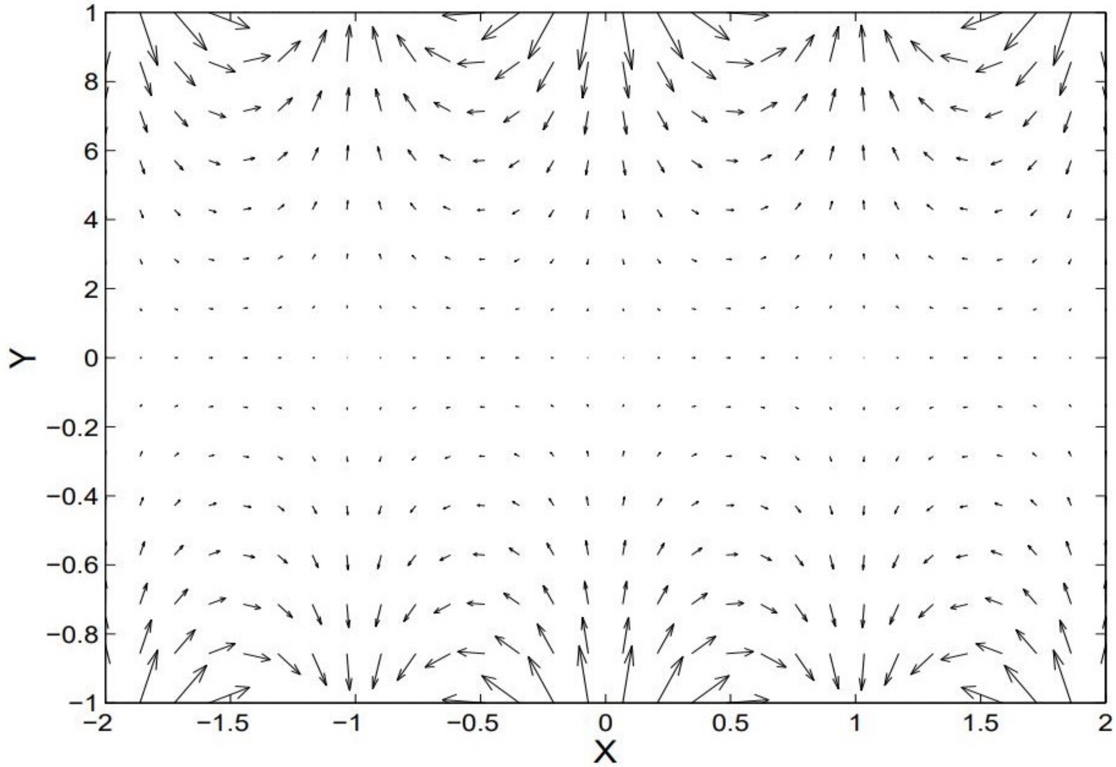

Figure 2: Difference in pressure gradients $\nabla P_N - \nabla P_D = 0.0508622\pi\cosh(\pi y)\sin(\pi x)\hat{\mathbf{i}} - 0.0508622\pi\sinh(\pi y)\cos(\pi x)\hat{\mathbf{j}}$

Finally, add the solutions to the two subproblems together.

P is indeterminate up to an additive constant so the resulting ∇P from both cases are considered instead. To do this, Kress and Montgomery rewrite the momentum and Poisson equations in dimensionless units and replace the kinematic viscosity with the reciprocal of the Reynold's number $Re = \frac{1}{\nu}\left(\frac{\langle \mathbf{v}^2\rangle}{k^2+\lambda^2}\right)^{\frac{1}{2}}$. Here, $\langle\cdot\rangle$ is the mean over a space containing one wavelength $2\pi/k$ in x from $y=-a$ to $y=a$.

In an article by Dietmar Rempfer [4], solutions are explicitly written. The Neumann pressure is

$$\begin{aligned} P_N = &-0.692376 - 0.61685\cos(2\lambda y) - 0.863371\cos(2\lambda y)\cosh(\frac{\pi y}{2}) - 0.0755258\cosh(\pi y) \\ &+ 3.30663\sin(\frac{\pi x}{2})\sinh(\frac{\pi y}{2}) + \cos(\pi x)[-1.6701 + 0.0124142\cos(\lambda y)\cosh(\frac{\pi y}{2}) \\ &+ 0.0489026\cosh(\pi y) + 0.491091\sin(\lambda y)\sinh(\frac{\pi y}{2})] \end{aligned}$$

and the Dirichlet pressure is

$$\begin{aligned} P_D = &-0.692376 - 0.61685\cos(2\lambda y) - 0.863371\cos(2\lambda y)\cosh(\frac{\pi y}{2}) - 0.0755258\cosh(\pi y) \\ &+ 3.30663\sin(\frac{\pi x}{2})\sinh(\frac{\pi y}{2}) + \cos(\pi x)[-1.6701 + 0.0124142\cos(\lambda y)\cosh(\frac{\pi y}{2}) \\ &+ 0.0997648\cosh(\pi y) + 0.491091\sin(\lambda y)\sinh(\frac{\pi y}{2})]. \end{aligned}$$

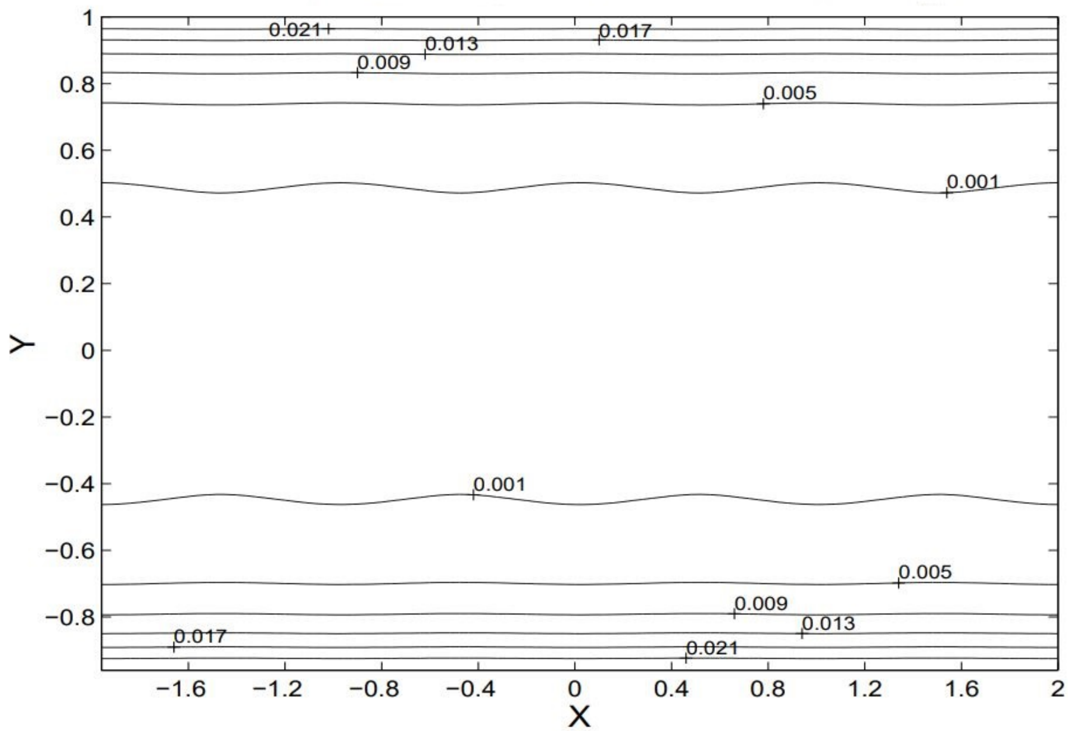

Figure 3: Normalized mean square pressure gradient difference

The resulting difference in pressure gradients is

$$\nabla P_N - \nabla P_D = 0.0508622\pi \cosh(\pi y)\sin(\pi x)\hat{\mathbf{i}} - 0.0508622\pi \sinh(\pi y)\cos(\pi x)\hat{\mathbf{j}}.$$

Figure 2 shows the difference in gradients between P_N, the pressure determined by the Neumann boundary conditions, and P_D, the pressure determined by the Dirichlet boundary conditions with $Re = 2293$ and $C_{k\lambda} = 5000$.

A fractional measure of the different pressure gradients, $\frac{(\nabla P_D - \nabla P_N)^2}{\langle(\nabla P_N)^2\rangle}$, is plotted in Figure 3. What's notable from the contour plot is the the scalar ratio disappears near the center and approaches a maximum at the wall. Additionally, the x component almost entirely disappears where the difference is largest. This is because the x component of $(\nabla P_N - \nabla P_D)^2$ is

$$\begin{aligned}
[0.0508622\pi \cosh(\pi y)\sin(\pi x)]^2 &= 0.0025870 \cosh^2(2ky)\cos^2(2kx) \\
&= 0.0025870 \frac{1+\cosh(4ky)}{2}\frac{1+\cos(4kx)}{2} \\
&= 0.0006467[1 + \cosh(4ky) - \cos(4kx) - \cosh(4ky)\cos(4kx)].
\end{aligned}$$

Here, $\cosh(4ky) - \cos(4kx)$ dominates the equation in a region where $ky \gtrsim 1$.

4 Possible Modification

We now look at the "Navier" boundary condition. Unlike the no-slip conditions considered earlier, there is some slip at the wall. The slip velocity can be written as $\Delta V = L_s\dot{\gamma}$ where ΔV is the slip velocity, L_s is a constant with the dimensions of length, and $\dot{\gamma}$ is the rate of shear at the wall.

An alternative that has worked in a different context [5], but has not been explored for this problem, is to replace the disappearance of the velocity at the wall with a wall friction term of the form $-\frac{\mathbf{v}}{\tau(x)}$. Here, $\tau(x)$ disappears in the interior of the fluid and attains large positive values near the wall. This terms allows the tangential component of the velocity to approach small values near the wall without making it vanish entirely. Thus $\mathbf{v}$ no longer satisfies the Dirichlet boundary condition and the pressure must be determined solely by Neumann boundary conditions.

Montgomery and Kress note that the velocity field considered does not lead to one which obeys the Navier boundary condition in positive time, as L_s will no longer be a constant but will vary sinusoidally with x. However, this paper is only concerned with initial conditions. Montgomery and Kress argue that the velocity field obtained from the considered stream function should be acceptable from the point of view of the Navier-Stokes or magnetohydrodynamic descriptions as it is thought to have all the relevant properties. In particular, the family of functions with the same x-periodicity is orthogonal and believed to be complete. In conclusion, the existence of velocity fields that are solenoidal and vanish at the walls which lead to equal Neumann and Dirichlet pressures is undetermined.

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