Week 4: Simple Linear Regression III

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Health Services Research Methods I
HSMP 7607
2019

Updated notes are here: https://clas.ucdenver.edu/marcelo-perraillon/teaching/health-services-research-methods-i-hsmp-7607

Outline

- Goodness of fit
- Confidence intervals
- Loose ends

Partitioning variance

- Recall from last class that we saw that the sum of the residual is zero, which implies that the predicted $\bar{\hat{y}}$ is the same as the observed $\bar{y}$, so always $\bar{\hat{y}} = \bar{y}$
- Also, for each observation i in our dataset the following is always true by definition of the residual:

  $y_i = \hat{y}_i + \hat{\epsilon}_i$, which is equivalent to

  $y_i = \hat{y}_i + (y_i - \hat{y}_i)$
- In words, an observed outcome value y_i is equal to its predicted value plus the residual
- We can do a little bit of algebra to go from this equality to something more interesting about the way we can interpret linear regression

Partitioning variance

- Subtract $\bar{y}$ from both sides of the equation and group terms:
  $(y_i - \bar{y}) = (\hat{y}_i - \bar{y}) + (y_i - \hat{y}_i)$
- Since $\bar{y} = \bar{\hat{y}}$:
  $(y_i - \bar{y}) = (\hat{y}_i - \bar{\hat{y}}) + (y_i - \hat{y}_i)$
- Stare at the last equation for a while. Looks familiar?
- Deviation from mean observed outcome = Deviation from fit for predicted values + Model residual
- The above equality can be expressed in terms of the sum of squares; the proof is messy. Your textbook skips several steps; see Wooldridge Chapter 2

Partitioning variance

- We need to define the following terms
- Total sum of squared deviations: $SST = \sum_{i=1}^{n}(y_i - \bar{y})^2$
- Sum of squares due to the regression: $SSR = \sum_{i=1}^{n}(\hat{y}_i - \bar{\hat{y}})^2$
- Sum of squared residuals (errors): $SSE = \sum_{i=1}^{n}(y_i - \hat{y})^2$
- We can then write $(y_i - \bar{y}) = (\hat{y}_i - \bar{\hat{y}}) + (y_i - \hat{y}_i)$ as
  $SST = SSR + SSE$
- SST/(n-1) is the sample variance of the outcome y
- SSR/(n-1) is the sample variance of the predicted values $\hat{y}$
- SSE/(n-1) is the sample variance of the residuals (but really, divided by $n - 2$)
- **Confusion alert**: in SSR, "R" stands for regression. In SSE, E stands for error, even though it should be "residual," not errors. Wooldridge uses SSE for "explained" or regression. And then Stata uses other labels...

Partitioning variance

- $SST = SSR + SSE$ is telling us that the observed variance of the outcome was partitioned into two parts, one that is explained by our model (SSR) and another that our model cannot explain (SSE)
- So we could then measure how good our model is by the ratio $\frac{SSR}{SST}$ (**goodness of fit**)
- In other words, the fraction of the total *observed variance* of the outcome that is *explained* by our model. That's the famous R^2: $R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$
- Another way: total observed variance = explained variance + unexplained variance

Understanding linear regression

- A newspaper article argued that Chicago's traffic is one of the most *unpredictable* in the nation. A commute that on average takes about 20 minutes can take 15, 40, 60, or even 120 minutes some days
- Say commuting time is the outcome y. What the article meant is that commuting time is highly **variable**. So $SST/(n-1)$ is high. Same as: the **sample variance or standard deviation** of y, s^2, is high. But it's **NOT** *unpredictable*
- You could develop a statistical model that explains y using weather (snow, rain) as predictor along with accidents, downtown events, day of week, time of day, and road work
- Once you estimate this model, SSE (unexplained/residual variance) will be **smaller** than a model without these predictors, and R^2 will be higher
- In other words, our model has **explained some of the observed variability** in commuting times. **I can't emphasize enough how important it is to understand these concepts**

Calculate using Stata

- Replicate R^2 in Stata output from Analysis of Variance (ANOVA) table (see **added** labels to the left)

```
reg colgpa hsgpa
                 sum of squares         mean SS
      Source |       SS           df       MS      Number of obs   =       141
-------------+----------------------------------   F(1, 139)       =     28.85
  (R)  Model |  3.33506006         1  3.33506006   Prob > F        =    0.0000
(E) Residual |  16.0710394       139  .115618988   R-squared       =    0.1719
-------------+----------------------------------   Adj R-squared   =    0.1659
 (T)   Total |  19.4060994       140  .138614996   Root MSE        =    .34003

------------------------------------------------------------------------------
      colgpa |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       hsgpa |   .4824346   .0898258     5.37   0.000      .304833    .6600362
       _cons |   1.415434   .3069376     4.61   0.000     .8085635    2.022304
------------------------------------------------------------------------------

. * replicate R^2
. di 3.33506006/19.4060994
.17185628

. di 1 - (16.0710394/19.4060994 )
.17185628
```

Calculate using Stata

- Replicate Mean Square Error, which is the unexplained variability of the model

```
reg colgpa hsgpa

      Source |       SS           df       MS      Number of obs   =       141
-------------+----------------------------------   F(1, 139)       =     28.85
       Model |  3.33506006         1  3.33506006   Prob > F        =    0.0000
    Residual |  16.0710394       139  .115618988   R-squared       =    0.1719
-------------+----------------------------------   Adj R-squared   =    0.1659
       Total |  19.4060994       140  .138614996   Root MSE        =    .34003

<... output omitted...>
. * Replicate root MSE.  This is the remaining *unexplained* variability
. di sqrt((16.0710394/139))
.34002792
* Compare to the variability without using hsgpa as a predictor (so just the variability of the outome, colgpa)
. sum colgpa
    Variable |        Obs        Mean    Std. Dev.       Min        Max
-------------+---------------------------------------------------------
      colgpa |        141    3.056738    .3723103        2.2          4
```

We can go from explained and unexplained standard deviation to R^2

- The trick is that the sum command is dividing by $n-1$ so we need to get the sum of squares. Root MSE divides by $n-2$

```
.  sum colgpa
 Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+---------------------------------------------------------
      colgpa |       141    3.056738    .3723103        2.2          4

* From the sum command, get SST
. di ((r(sd)^2)*(141-1))
19.406099

* From the regression output, use the model unexplained, Root MSE to get SSR
. di (((.34002792)^2)*139)
16.071039

* Combine so you can replicate R^2
. di 1 - (((.34002792)^2)*139) / ((r(sd)^2)*140)
.17185629
```

Another way of understanding R^2

- Again, one way of defining R^2 is that it is the ratio of the variation explained by the model (SSR) to the total variability (SST)
- It turns out that R^2 is also the square of the correlation between the observed y and its model-predicted values: $cor(y, \hat{y})^2$

```
 qui reg colgpa hsgpa
predict colhat
(option xb assumed; fitted values)

. corr colhat colgpa
(obs=141)

             |   colhat   colgpa
-------------+------------------
      colhat |   1.0000
      colgpa |   0.4146   1.0000

. di  0.4146^2
.17189316
```

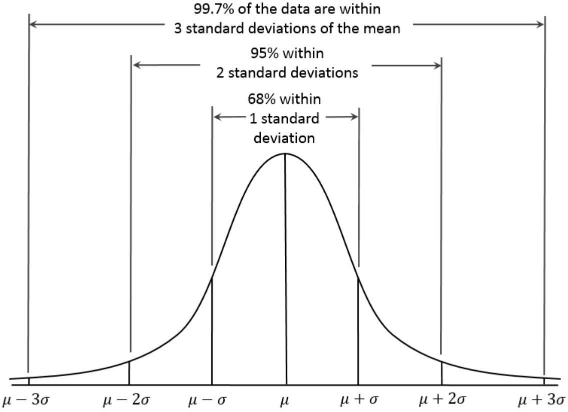
- The better our model is at predicting y the higher R^2 will be

Confidence Intervals

- We want to build a confidence interval for $\hat{\beta}$
- The proper interpretation of a confidence interval in the frequentist approach is that if we repeated the experiment many times, about x% percent of the time the value of β would be within the confidence interval
- By convention, we build 95% confidence intervals, which implies $\alpha = 0.05$
- Intuitively, we need to know the distribution of $\hat{\beta}$ and its precision, the standard error. To derive these, we need to assume ϵ distributes $N(0, \sigma^2)$ iid
- A formula for the confidence interval of $\hat{\beta}_i$ is: $\hat{\beta}_i \pm t_{(n-2,\alpha/2)} se(\hat{\beta}_i)$
- We saw that $t_{(n-2,\alpha/2)}$ in the context of Wald tests. In the normal, it's 1.98

Confidence Intervals

- We use t-student but remember that when *n* is large (larger than about 120) the t distribution approximates a normal distribution
- Recall this graph from stats 101 and Wikipedia :



Direct relationship between statistical tests and confidence intervals

- Confidence intervals and statistical tests are closely related

```
 reg colgpa hsgpa
------------------------------------------------------------------------------
      colgpa |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       hsgpa |   .4824346   .0898258     5.37   0.000      .304833    .6600362
       _cons |   1.415434   .3069376     4.61   0.000     .8085635    2.022304
------------------------------------------------------------------------------
. test hsgpa = .304833
 ( 1)  hsgpa = .304833
       F(  1,   139) =    3.91
            Prob > F =    0.0500
. test hsgpa = .31
 ( 1)  hsgpa = .31
       F(  1,   139) =    3.69
            Prob > F =    0.0570
. test hsgpa = .29
 ( 1)  hsgpa = .29
       F(  1,   139) =    4.59
            Prob > F =    0.0339
```

- **Remember this**: If the number θ_0 in null $H_0 : \beta_j = \theta_0$ is within $95\% CI$, we won't reject a null for that value; if the number is outside CI, we will reject

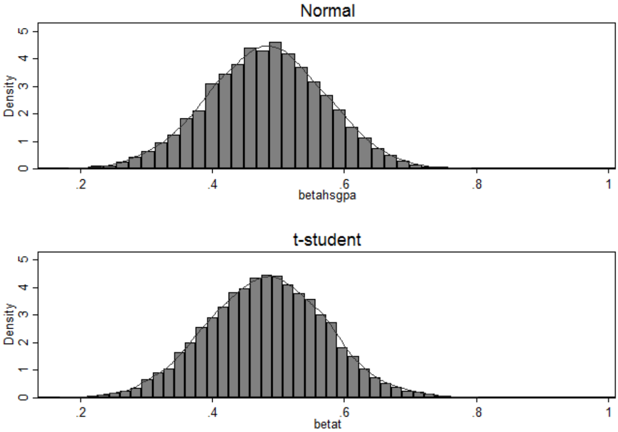
As usual, simulations are awesome

```
 * simulate 9000 observations
 set obs 9000
gen betahsgpa = rnormal(.4824346,.0898258)
 sum
   Variable |        Obs        Mean    Std. Dev.       Min        Max
-------------+---------------------------------------------------------
  betahsgpa |      9,000    .4826007    .0900429   .1583372   .9124259
 * count the number of times beta is within the confidence interval
 count if betahsgpa >= .304833 & betahsgpa <= .6600362
  8,552
 di 8552/9000
.95022222
 * Do the same with a t-student
. gen zt = rt(139)
. * By default, Stata simulate a standard t, mean zero and sd of 1
sum zt
   Variable |        Obs        Mean    Std. Dev.       Min        Max
-------------+---------------------------------------------------------
         zt |      9,000    .0019048    .9972998  -3.567863   3.574553
. * Need to retransform
 gen betat = .0898258*zt + .4824346
 sum betat
   Variable |        Obs        Mean    Std. Dev.       Min        Max
-------------+---------------------------------------------------------
      betat |      9,000    .4826057    .0895833   .1619485   .8035216
 count if betat >= .304833 & betat <= .6600362
  8,576
. di 8576/9000
.95288889
```

What just happened?

- We estimated a parameter $\hat{\beta}$ and its standard error $\sqrt{var(\hat{\beta})}$
- Because of assumptions about ϵ distributing $N(0, \sigma^2)$ we know that asymptotically $\hat{\beta}$ distributes normal (but the Wald test distributes t-student)
- That's all we need to calculate confidence intervals and hypotheses tests about the true β in the population
- Recall that a probability distribution function describes the values a random variable can take and the probability of those values. So we can make statements about the probability of the parameter taking certain values or being within an interval if we know the distribution of the parameter
- We also know that the t-student converges to a normal for samples larger than about 120, so we could just use the normal distribution

Distributions



- Note the slightly fatter tails of the t-student

We can do more

- Other confidence intervals?

```
centile betahsgpa, centile(2.5(5)97.5)
```

Variable	Obs	Percentile	Centile	-- Binom. Interp. -- [95% Conf.	Interval]
betahsgpa	9,000	2.5	.3028123	.2985525	.3087895
		7.5	.3527713	.3494937	.3565182
		12.5	.3789864	.375991	.3828722
		17.5	.3990816	.3966145	.4013201
		22.5	.4146384	.4117871	.4172412
		27.5	.428773	.4263326	.4314667
		32.5	.4425001	.440095	.4447364
		37.5	.4546066	.4523425	.4566871
		42.5	.466314	.4637115	.4679226
		47.5	.4773481	.4747118	.4795888
		52.5	.4890074	.486511	.4909871
		57.5	.4991535	.4969174	.5015363
		62.5	.5109765	.5086741	.5133994
		67.5	.5230142	.5210292	.5253365
		72.5	.5359863	.5334756	.5384137
		77.5	.5504502	.5473233	.5534496
		82.5	.5673794	.5640415	.5700424
		87.5	.5862278	.5831443	.589366
		92.5	.6109914	.6076231	.6150107
		97.5	.6565534	.6518573	.6633566

- Note how close the 2.5 and 97.5 percentiles follow the reg CI above

Even more

- What is the probability that the coefficient for hsgpa is greater than 0.4? Greater than 0.8?

```
. count if betahsgpa >0.4
  7,394
. di 7394/9000
.81822222
 count if betahsgpa >0.8
  2
 di 2/9000
.00022222
```

- Fairly likely and fairly unlikely, respectively (look at distribution)
- We have no evidence that the coefficient will be remotely close to zero so no surprise about the p-value in the regression output
- **Caution**: We only have one predictor/covariate here. With more, there is a correlation between $\hat{\beta}_j$. They distribute multivariate normal with a variance-covariance matrix. We will see examples

What about Type II error?

- The other error we can make when testing hypotheses is **Type II error**
- Type II: failing to reject the null when in fact is false
- We saw that the power of a test is 1-P(Type II)
- When are we going to fail to reject the null even if it's false? Intuitively, when the confidence interval is wide
- With a very wide CI, more values are going to be within the CI so they won't get rejected; they are likely to happen at $\alpha = 0.05$
- And when is the confidence interval going to be wide? Look at the formula for CI: when *t*() or *se*() are larger. Both **depend on sample size**
- When doing **power analysis**, we're mainly concerned about determining the sample size we need to avoid Type II error

Loose ends

- We have only one thing left to explain and replicate from the regression output. What is that F test?
- In the grades example:

```
      Source |       SS           df       MS      Number of obs   =       141
-------------+----------------------------------   F(1, 139)       =     28.85
       Model |  3.33506006         1  3.33506006   Prob > F        =    0.0000
    Residual |  16.0710394       139  .115618988   R-squared       =    0.1719
-------------+----------------------------------   Adj R-squared   =    0.1659
       Total |  19.4060994       140  .138614996   Root MSE        =    .34003
```

- That's a test of the overall validity of the model. The null is that $\beta_1 = ... = \beta_j = 0$. Here, only one, so $H_o : \beta_1 = 0$
- It's the ratio $MSR/MSE = 3.33506006/.115618988 = 28.845263$ (so regression/model to residual). As you know by now, it is F because it is the ratio of two chi-squares
- Once we cover maximum likelihood we we will see another more general approach to compare models, the **likelihood ratio test**
- Both test are (**asymptotically**) equivalent

Preview

- F test versus LRT (I chose male as predictor so the p-value is not close to zero)

```
reg colgpa male

      Source |       SS           df       MS      Number of obs   =       141
-------------+----------------------------------   F(1, 139)       =      0.82
       Model |  .113594273         1  .113594273   Prob > F        =    0.3672
    Residual |  19.2925052       139  .138795001   R-squared       =    0.0059
-------------+----------------------------------   Adj R-squared   =   -0.0013
       Total |  19.4060994       140  .138614996   Root MSE        =    .37255
<... output omitted...>
------------------------------------------------------------------------------
      colgpa |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        male |  -.0568374   .0628265    -0.90   0.367    -.1810567    .0673818
       _cons |   3.086567   .0455145    67.82   0.000     2.996577    3.176557
------------------------------------------------------------------------------
est sto m1
qui reg colgpa
est sto m2
. lrtest m1 m2
Likelihood-ratio test                                 LR chi2(1)  =      0.83
(Assumption: m2 nested in m1)                         Prob > chi2 =    0.3629
```

- **Three tests in one**, all close to 0.36. Since we only have one predictor/covariate, the null of the overall F test is that $\beta_1 = 0$, same as Wald test

Summary

- Linear regression can be thought of as partitioning the variance into two components, explained and unexplained
- We can measure the goodness of fit of a model based on the comparison of these variances
- **Be carefully about the context**. We are talking about a linear model with ϵ $N(0, \sigma^2)$ and iid
- Not the same in other type of models but the main ideas are valid
- Once we know the asymptotic distribution of a parameter and its standard deviation (i.e. standar error) we have all we need to test hypotheses and build CIs